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Low-dimensional model of spatial shear layers

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The aim of this work is to develop nonlinear low-dimensional models to describe vortex dynamics in spatially developing shear layers with periodicity in time. By allowing a free variable \( g(x) \) to dynamically describe downstream thickness spreading, we are able to obtain basis functions in a scaled reference frame and construct effective models with only a few modes in the new space. To apply this modified version of proper orthogonal decomposition (POD)/Galerkin projection, we first scale the flow along \( y \) dynamically to match a template function as it is developing downstream. In the scaled space, the first POD mode can capture more than 80% energy for each frequency. However, to construct a Galerkin model, the second POD mode plays a critical role and needs to be included. Finally, a reconstruction equation for the scaling variable \( g \) is derived to relate the scaled space to physical space, where downstream spreading of shear thickness occurs. Using only two POD modes at each frequency, our models capture the basic dynamics of shear layers, such as vortex roll-up (from a one-frequency model) and vortex-merging (from a two-frequency model). When arbitrary excitation at different harmonics is added to the model, we can clearly observe the promoting or delaying/eliminating vortex merging events as a result of mode competition, which is commonly demonstrated in experiments and numerical simulations of shear layers. © 2012 American Institute of Physics. [doi:10.1063/1.3678016]

I. INTRODUCTION

Shear layers are often studied as model flows for their simplicity. Behaviors of temporally developing (TD) and spatially developing (SD) shear layers have been studied for several decades from theoretical, numerical, and experimental perspectives.\(^1\)–\(^5\) However, shear flows are still too complex to directly apply dynamic systems and control theories which have been widely used to analyze and understand many simple mechanical systems. Though low-dimensional models have been proposed\(^3\),\(^5\) and succeeded in some applications, they are mainly phenomenological. The goal here is to seek an efficient definition of base modes to represent the flow physics and then develop low-dimensional models from the direct projection of first-principle governing equations onto these modes.

Based on a combination of proper orthogonal decomposition (POD)/Galerkin projection method and symmetry reduction ideas introduced from geometric mechanics, Wei and Rowley recently developed low-dimensional models for TD free shear layers.\(^6\) A scaling variable \( g(t) \) was introduced to factor out the thickness growth so that similar vortex structures at different thicknesses can be uniformly described. It should be pointed out that the scaling factor was introduced and calculated simultaneously without \textit{a priori} knowledge of self-similarity. Similar techniques have been used for traveling solutions\(^7\) and self-similar solutions.\(^8\)

However, when the same idea was applied on SD shear layers, many new challenges became apparent from sometimes critical difference between temporal and spatial development.\(^9\) First,
since the self-similarity happens in the stream-wise ($x$) direction for SD flows, a scaling factor $g(x)$ as a function of $x$ becomes a natural choice for symmetry reduction. Second, the periodicity can only exist in time and is normally introduced through extra constraints (e.g., external forcing). Last, the derived low-order model for SD shear layers is an ordinary differential equation evolving in space instead of time. The major modeling difficulty caused by the above differences is due to the fact that the Navier-Stokes equations are parabolic in time but not parabolic in space. However, there is an obvious remedy, which is to parabolize the Navier-Stokes equations. Such parabolization has been explored for decades for efficient computation of spatially developing flows. The earliest example traces back to Prandtl’s famous boundary-layer equations, which are strictly parabolic along the streamwise direction. Later, similar concepts were applied to a wider range of problems by solving the parabolized Navier-Stokes (PNS) equations. From an instability analysis perspective, the parabolization of the Navier-Stokes equations leads to the parabolized stability equations (PSE), which extend traditional linear instability analysis to cover “non-parallel” flows with slow variation downstream. It is worth noting that both PNS and PSE are not strictly parabolic because of the presence of the pressure terms, which introduces limitations in both approaches. One common practice is to remove the stream-wise pressure gradient to break the ellipticity. In our work, to further simplify the model, we push the approximation further by removing the pressure gradient entirely from all equations. The simplification is justified by introducing the conditions of “thin layer” and uniform far-field flow, which are usually satisfied in boundary-layer type flows including SD shear layers. The model can, therefore, be derived solely from momentum equations. The decoupling from the continuity equation results in a slow shift of total mass, which, however, has not shown a fundamental effect on vortex dynamics being modeled here.

For spatially developing shear layers, mode competition has been studied experimentally and numerically as a trademark problem. It appears most simply when single-frequency excitation is introduced to a shear layer with broadband background noise. The artificial excitation promotes the dynamics of its own frequency, and, at the same time, suppresses/delays the appearance of dynamics at other frequencies. With or without being promoted by external excitation, such nonlinear interaction between the harmonics and subharmonics always exists and often plays an essential role in mixing layer dynamics. In mixing layers, the forcing frequency and forcing amplitude have significant effects on vortex merging, such as the number of vortices in each merging and the location of the merging. Moreover, Ho and Huang defined the merging/pairing location as the position where the subharmonic frequency saturates and reaches its peak. They also explained that exciting the fundamental frequency suppresses the subharmonic growth and delays the vortex pairing, while exciting the subharmonic growth promotes the vortex pairing. In the present work, mode competition is investigated to test the robustness of our model and also serves as an application.

The paper is arranged as follows. Direct numerical simulation (DNS) of SD shear layers is described in Sec. II. Equation parabolization for modeling and an extension with external body forces is discussed in Sec. III. Section IV describes the entire methodology of low-dimensional modeling for SD shear layers. The results with further comparison and discussions are in Sec. V. Finally, the conclusions are summarized at the end in Sec. VI.

II. SIMULATION OF SPATIALLY DEVELOPING SHEAR LAYERS

The flow considered in this paper is a two-dimensional, spatially developing, free shear layer as shown schematically in Figure 1. The Reynolds number of the flow is $Re = 200$, assuming the characteristic length equal to the initial vorticity thickness $\delta_{\omega} = \Delta U/|du/dy|_{\text{max}}$ at the entrance and the characteristic velocity equal to the maximum velocity change across the shear layer $\Delta U$. All values are non-dimensionalized by the same characteristic values for the rest of the paper. Although the modeling will be based on the equations derived from incompressible Navier-Stokes equations, the simulation itself solves the fully compressible Navier-Stokes equations at low Mach number, using a code that has been validated in previous work. The velocity divergence induced by weak compressibility here is small and, therefore, has negligible impact on current modeling if compressibility-specific features (e.g., acoustics) are not considered. The usage of
weakly compressible data for incompressible modeling is common and has been successfully applied in our previous work on temporally developing shear layers.6

The flow is simulated in a domain extending 200 in the stream-wise direction, \( x \), and out to \( 680 \) in the transverse direction, \( y \), from the mixing layer. Extra buffer areas with a length 20 in \( y \) to the top and bottom and a length 60 in \( x \) to the left and right are applied in computation. To make the flow periodic in time, we apply a body force excitation in a small box area \( 5 < x < 15 \), \( -5 < y < 5 \) to trigger the instability. Two excitation frequencies \( k = 1 \) and \( k = 2 \) are picked based on non-dimensional time period \( T = 38.4 \). The values are chosen such that, at the initial stage, the frequency \( k = 2 \) is near the most unstable frequency predicted by linear instability, and \( k = 1 \) is more stable as the thickness increases downstream.22

III. PARABOLIZATION OF GOVERNING EQUATIONS

The governing equations for two-dimensional incompressible flow are

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= - \frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= - \frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).
\end{align*}
\] (1)

Mathematically, the elliptic behavior in space arises in two ways through: (1) the Laplacian of velocity in viscous terms and (2) the pressure terms implied in a Poisson equation. The first one can be removed in a relatively easy way by assuming a thin layer,10,11 which ignores all second-order stream-wise derivatives. The same approximation exists in all previous efforts for the parabolization of the Navier-Stokes equations.3,12–14 A common approach15 to remove the second type of ellipticity is to remove the pressure gradient downstream \( (\partial p/\partial x) \). To further simplify the model, applying the thin layer assumption with an outer flow \( U(x) \) being a function of \( x \) only can render all the pressure gradients negligible.11 Extra caution is taken here to avoid further simplification which may artificially alter instability features.23,24 Thus, the reduced parabolic equations for later modeling are

\[
\begin{align*}
\frac{u}{\partial x} = -v \frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}, \\
\frac{\partial v}{\partial x} = -v \frac{\partial v}{\partial y} - \frac{\partial v}{\partial t} + \frac{1}{Re} \frac{\partial^2 v}{\partial y^2},
\end{align*}
\] (2)
where terms are rearranged for convenience. Pressure can be viewed as a Lagrange multiplier in the momentum equations to re-enforce the continuity equation (mass conservation). The simplification in Eq. (2) apparently releases this constraint, that is, the system mass can drift. In our previous modeling for TD shear flow, we noticed that the model without mass conservation\textsuperscript{25} is less accurate than the model with mass conservation.\textsuperscript{6} However, the mass drift is slow and does not change the fundamental dynamics, and the model without mass conservation, in its simple form and of low order, can still capture basic vortex dynamics. For SD shear flow, a low-order model based on Eq. (2) is expected to represent all basic vortex dynamics. We want to emphasize that although Galerkin projection is based on the parabolized equation (2), the numerical simulation of the flow field uses the complete Navier-Stokes equations which assure the accuracy of the basis functions (modes).

For flow control studies, excitation is normally introduced and modeled as body-force terms in momentum equations. Following the same idea as above, the Navier-Stokes equations with body-force excitation can be parabolized to

\begin{align}
\frac{\partial u}{\partial x} &= -v \frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} + \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial y^2} + f_u(y, t), \\
\frac{\partial v}{\partial x} &= -v \frac{\partial v}{\partial y} - \frac{\partial v}{\partial t} + \frac{1}{\text{Re}} \frac{\partial^2 v}{\partial y^2} + f_v(y, t),
\end{align}

where, for simplicity, we assume excitation forces \( f_u \) and \( f_v \) are functions of \( y \) and \( t \) only. Based on Eq. (3), we can extend the low-dimensional models to include the effects from external controls. Such extension serves as both an application and a test for model robustness as shown later.

### IV. LOW-DIMENSIONAL MODELS

#### A. Scaling the flow dynamically

A common approach for low-dimensional modeling is to project the governing equations onto a fixed set of basis functions, which are determined mathematically (i.e., Fourier modes) or empirically (i.e., POD modes). Since the shear layer thickness is slowly spreading downstream, we consider basis functions that can scale in \( y \) direction to accommodate the slow variation. A similar idea has been successfully applied to temporally developing periodic shear flows.\textsuperscript{6,25} The main difference of the current scaling is that the scaling function becomes a function of \( x \) instead of \( t \) by its spatially developing nature. If the velocity vector is defined by \( \mathbf{q} = (u \ v)^T \), a scaled variable \( \tilde{\mathbf{q}} \) can be introduced by

\[ \tilde{\mathbf{q}}(x, y, t) = \mathbf{q}(x, g(x)y, t), \]

where \( g(x) \) is a scaling factor to be determined. The purpose of introducing \( g(x) \) is to factor out the mean flow development so that the flow dynamics can be represented by fewer modes. Consequently, \( g(x) \) is defined here to line up a scaled solution \( \tilde{\mathbf{q}} \) that best matches a pre-selected template function. The initial shear flow profile \( \mathbf{q}_0 = (u_0 \ v_0)^T \) can be a natural choice for this template, where

\[ u_0 = U_1 + \frac{U_2 - U_1}{2} (1 + \tanh(2y)), \quad v_0 = 0. \]

It should be noted that the only non-zero component in the template is \( u_0 \). The scaling factor \( g(x) \) is, therefore, defined by

\[ g(x) = \arg \min_g ||u^T \left( x, \frac{y}{g}, t \right) - u_0^T(y)||^2, \]

where \( || \cdot ||^2 \) is a \( L^2 \) norm defined by integrating over \( y \) and over a single time period. A new thickness \( \delta_x \) can be defined by \( g(x) \) as
\[ \delta_g = \frac{1}{g(x)}, \quad (7) \]

which can be used to measure the shear layer spreading as an alternative to vorticity thickness or momentum thickness. In fact, all three thicknesses represent the shear layer spreading in a qualitatively similar way. The condition for \( \tilde{u}(x, y, t) := u(x, y/g, t) \) to enforce the best match to the template is

\[ \frac{d}{ds} \left| \left| \tilde{u}^2(x, y, t) - u_0^2(h(s)y) \right| \right|^2 = 0, \quad (8) \]

where \( h(s) \) is any curve in \( \mathbb{R}^+ \) with \( h(0) = 1 \), and the same norm on the space of functions of \( (y, t) \) is used, that is, \( h = 1 \) is a local minimum of the error norm above. Here, \( \tilde{u}^2 \) is used instead of \( \tilde{u} \) as in our previous work for both TD (Ref. 6) and SD (Ref. 9) shear flows. This choice of template function is based on both physical and mathematical considerations. Physically, to choose \( \tilde{u} \) as the template is analogous to the definition of displacement thickness, and to choose \( \tilde{u}^2 \) as the template is analogous to the definition of momentum thickness. The whole idea of symmetry reduction here is to separate the slow growth of shear layer thickness, so that, POD modes can be more efficiently obtained in the new scaled space. To physically represent the shear layer thickness, both \( \tilde{u} \) and \( \tilde{u}^2 \) are good choices for the template. However, mathematically, the minimization problem using \( \tilde{u} \) as the template eventually leads to the derivative \( \partial u / \partial t \) for TD flows and \( \partial u / \partial x \) for SD flows. In the Navier-Stokes equations, the term \( \partial u / \partial x \) only exists as part of the convective term, which is more naturally presented by using \( \tilde{u}^2 \) as the template such that \( \partial (u^2) / \partial x = 2u \partial u / \partial x \). So, mathematically, \( \tilde{u} \) is a better choice for TD flows, and \( \tilde{u}^2 \) is a better choice for SD flows. Therefore, the new choice here simplifies the modeling of the convective term \( u \partial u / \partial x \) for SD flows and results in a simpler and more accurate equation for \( g(x) \). Such mathematical convenience can be shown more clearly in the later derivation of \( g(x) \). This choice of Eq. (8) yields

\[ -2 \left\langle \frac{d}{ds} \left| \left| \tilde{u}^2(h(s)y), \tilde{u}^2(x, y, t) - u_0^2(y) \right| \right|^2 = 0, \right. \]

which becomes

\[ \left\langle yu_0 \frac{\partial u_0}{\partial y}, \tilde{u}^2 - u_0^2 \right\rangle = 0. \quad (9) \]

Geometrically, the result in Eq. (9) means that the set of all such functions \( \tilde{u}^2 \) that are scaled so that they most closely match the template \( u_0^2 \) is an affine space through \( u_0^2 \) and orthogonal to \( yu_0 \partial u_0 / \partial y \).

**B. Equations in scaled space**

The parabolized equations consist of flow variables at \((y, t)\) marching in \(x\) and can be regarded as a dynamical system evolving on a function space \( H \). Thus, \( q(x) \in H \) is a snapshot of the entire flow at location \( x \), and Eq. (2) may be written as

\[ A \frac{\partial q(x)}{\partial x} = h(q(x)), \quad (10) \]

where

\[ A = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}, \quad (11) \]

to form the left-hand-side of Eq. (2), and \( h \) is a differential operator on \( H \). Here, matrix \( A \) can also be written in scaled variables as

\[ \tilde{A} = \begin{bmatrix} \tilde{u} & 0 \\ 0 & \tilde{u} \end{bmatrix}, \quad (12) \]
and the same scaling holds as
\[ A(x, y, t) = \hat{A}(x, g(x)y, t). \] (13)

If we introduce the scaling operator \( S_g : H \rightarrow H \), defined by
\[ S_g[q(y, t)] = q(gy, t), \quad \forall g \in \mathbb{R}^+, \] (14)
then the scaling equation (4) becomes \( q = S_g[\tilde{q}] \), Eq. (13) becomes \( A = S_g[\hat{A}] \), and the governing equation (10) may be written as
\[ S_g[\hat{A}(x, y, t)] \frac{\partial}{\partial x} S_g[q(x, y, t)] = h(S_g[q(x, y, t)]). \] (15)

Since
\[ \frac{\partial}{\partial x} S_g[q(x, y, t)] = \frac{\partial}{\partial x} \tilde{q}(x, g(x)y, t) \]
\[ = \frac{\partial \tilde{q}}{\partial x}(x, gy, t) + \dot{g} y \frac{\partial \tilde{q}}{\partial y}(x, gy, t) \]
\[ = S_g \left[ \frac{\partial \tilde{q}}{\partial x} \right] + \frac{\dot{g}}{g} S_g \left[ y \frac{\partial \tilde{q}}{\partial y} \right], \] (16)

Eq. (15) becomes
\[ S_g[\hat{A}(x, y, t)] \frac{\partial}{\partial x} S_g[q(x, y, t)] = h(S_g[q(x, y, t)]). \] (17)

If we define \( S_{1/g} \) as an inverse mapping of \( S_g \) such that \( h_g(\tilde{q}) = S_{1/g} h(S_g[q]) \), applying the inverse mapping to above equation, we have the governing equations in scaled space,
\[ \hat{A} \frac{\partial \tilde{q}}{\partial x} = h_g(\tilde{q}) - \frac{\dot{g}}{g} \frac{\partial \tilde{q}}{\partial y}. \] (18)

Equation (18) can be written separately in variables \( \tilde{u} \) and \( \tilde{v} \) as
\[ \tilde{u} \frac{\partial \tilde{u}}{\partial x} = h_g \tilde{v} - \frac{\dot{g}}{g} \frac{\partial \tilde{u}}{\partial y}, \]
\[ \tilde{v} \frac{\partial \tilde{v}}{\partial x} = h_g^2 \frac{\partial \tilde{u}}{\partial y}. \] (19)

However, these equations alone are not sufficient to evolve the dynamics without the knowledge of \( g(x) \).

C. Equation for scaling variable

In this section, the evolution equation for \( g(x) \) will be derived to close the system. Differentiating the constraint of Eq. (9) along \( x \), we have
\[ \left( y \frac{\partial u_0}{\partial y}, \frac{\partial \tilde{u}}{\partial x} \right) = 0. \] (20)

Using the \( x \)-momentum equation (19),
\[ \frac{\partial \tilde{u}}{\partial x} = h_g \frac{\partial \tilde{u}}{\partial y}, \] (21)
so that, we have
which becomes
\[
\dot{\frac{g}{\hat{g}}} = \frac{\left\langle \frac{h^1}{\hat{g}}, y_0 \frac{\partial u_0}{\partial y} \right\rangle}{\left\langle y_\hat{u} \frac{\partial u}{\partial y}, y_0 \frac{\partial u_0}{\partial y} \right\rangle}.
\] (23)

Altogether, Eq. (18) for \( \dot{\mathbf{q}} \) and Eq. (23) for \( \dot{g} \) define the system evolution in a scaled space without slow variation from the shear layer thickness.

D. Galerkin projection

Before we implement the projection, it is necessary to group the equations’ right-hand-side into nonlinear terms \( N \), linear terms \( L \), and body-force terms \( f_q \) which only appear when external control is considered, as
\[
h(\mathbf{q}) = N(\mathbf{q}, \mathbf{q}) + L(\mathbf{q}) + f_q,
\] (24)
where
\[
N(\mathbf{q}, \mathbf{q}) = \left[ \begin{array}{c} -v \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{array} \right], \quad L(\mathbf{q}) = \left[ \begin{array}{c} \frac{\partial u}{\partial t} + \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial v}{\partial t} + \frac{1}{Re} \frac{\partial^2 v}{\partial y^2} \end{array} \right], \quad f_q = \left[ \begin{array}{c} f_u \\ f_v \end{array} \right].
\] (25)

The corresponding terms in the scaled space are
\[
h_{\hat{g}}(\mathbf{\hat{q}}) = N_{\hat{g}}(\mathbf{\hat{q}}, \mathbf{\hat{q}}) + L_{\hat{g}}(\mathbf{\hat{q}}) + f_{\hat{q}},
\] (26)
where
\[
N_{\hat{g}}(\mathbf{\hat{q}}, \mathbf{\hat{q}}) = \left[ \begin{array}{c} -\hat{v} \frac{\partial \hat{u}}{\partial y} \\ -\hat{v} \frac{\partial \hat{v}}{\partial y} \end{array} \right], \quad L_{\hat{g}}(\mathbf{\hat{q}}) = \left[ \begin{array}{c} -\frac{\partial \hat{u}}{\partial t} + \frac{1}{Re} \hat{g} \frac{\partial^2 \hat{u}}{\partial y^2} \\ -\frac{\partial \hat{v}}{\partial t} + \frac{1}{Re} \hat{g} \frac{\partial^2 \hat{v}}{\partial y^2} \end{array} \right], \quad f_{\hat{q}} = \left[ \begin{array}{c} f_{\hat{u}} \\ f_{\hat{v}} \end{array} \right].
\] (27)

We can then expand \( \mathbf{\hat{q}} \) and \( f_{\hat{q}} \) in their base functions as
\[
\mathbf{\hat{q}} = \mathbf{\hat{q}}_0(y) + \sum_{k=-\infty}^{+\infty} \sum_{n=0}^{+\infty} a_{k,n}(x) \Phi_{k,n}(t,y),
\] (28)
\[
f_{\hat{q}} = \sum_{k=-\infty}^{+\infty} \sum_{n=0}^{+\infty} A_{k,n}(t,y),
\] (29)
where
\[
\Phi_{k,n}(t,y) = e^{2\pi i k T} \phi_{k,n}(y).
\] (30)

Here, \( k \) is the frequency, \( T \) is the time period, \( a_{k,n}(x) \) are the POD coefficients, \( \phi_{k,n} = (\hat{u}_{k,n}, \hat{v}_{k,n})^T \) is the \( n \)th POD mode for frequency \( k \), and \( A_{k,n} \) are the control force coefficients. For simplicity, external control is arbitrarily assumed to be a combination of POD modes at different frequencies. The frequency and strength of external excitations can then be adjusted easily by altering the value of \( A_{k,n} \). The standard vector inner product is used in the POD calculation and the later Galerkin projection,\(^{25}\) as
\[ \langle \mathbf{a}, \mathbf{b} \rangle = \int_{\Omega} (\tilde{a}_1 \tilde{b}_1 + \tilde{a}_2 \tilde{b}_2) dt \, dy, \]  

(31)

where \( \tilde{\cdot} \) indicates a quantity in the scaled space, and the time integration is actually absorbed during the computation after substituting in each individual Fourier mode. The energy of each POD mode \((k, n)\) is quantified by

\[ \lambda_{k,n} = \langle \mathbf{q} - \mathbf{q}_0, \Phi_{k,n} \rangle^2 = |a_{k,n}|^2, \]

(32)

where \( \tilde{\cdot} \) denotes a streamwise spatial average.

We start with a simple case retaining only frequencies \( k = \pm 1 \), and the first two POD modes \( n = 1 \) and \( n = 2 \) for each frequency. The summation is then an approximation of the original \( \mathbf{q} \). The notation \( \mathbf{q} \) is retained for the finite sum in Eq. (28), and since \( \mathbf{q} \) must be real, the additional constraint,

\[ a_{1,1} \Phi_{1,1} + a_{1,2} \Phi_{1,2} = a^*_{-1,1} \Phi^*_{-1,1} + a^*_{-1,2} \Phi^*_{-1,2}, \]

(33)

permits further simplification of the equations that follow.

To obtain the equations for coefficients \( a_{1,1}(x) \) and \( a_{1,2}(x) \), the governing equation (18) are projected onto modes \( \Phi_{1,1} \) and \( \Phi_{1,2} \). Eventually, the spatial evolution equation for coefficient vector \( \mathbf{a} = (a_{1,1} \ a_{1,2})^T \) is

\[ \mathbf{B} \mathbf{a} = \left( \mathbf{gC} + \Lambda + \frac{1}{Re} g^2 \mathbf{D} + \frac{\dot{g}}{g} \mathbf{E} \right) \mathbf{a} + \mathbf{F}. \]

(34)

Matrices \( \mathbf{B} \), \( \mathbf{C} \), \( \Lambda \), \( \mathbf{D} \), \( \mathbf{E} \), and \( \mathbf{F} \) are defined by

\[
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix},
\begin{bmatrix}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{bmatrix},
\begin{bmatrix}
  c_{13} & 0 \\
  0 & c_{23}
\end{bmatrix},
\begin{bmatrix}
  d_{11} & d_{12} \\
  d_{21} & d_{22}
\end{bmatrix},
\begin{bmatrix}
  e_{11} & e_{12} \\
  e_{21} & e_{22}
\end{bmatrix},
\begin{bmatrix}
  A_{11} \\
  A_{12}
\end{bmatrix},
\]

(35)

where all coefficients in the matrices are well-defined and listed in Appendix A with some detailed derivation. To close the system, the spatial evolution for thickness adjustment \( g(x) \) is needed and can be obtained from the scaling relation (23) with the same two modes being retained,

\[ \dot{g} = \frac{g(c_{01} a_{1,1} a_{1,1}^* + c_{02} a_{1,1} a_{1,2}^* + c_{03} a_{1,2} a_{1,1}^* + c_{04} a_{1,2} a_{1,2}^*) g^3 + \frac{1}{Re} \int_{\Omega} \mathbf{b}_0 g^3}{b_0 + b_{01} a_{1,1} a_{1,1}^* + b_{02} a_{1,1} a_{1,2}^* + b_{03} a_{1,2} a_{1,1}^* + b_{04} a_{1,2} a_{1,2}^*}, \]

(36)

where all coefficients are also defined in Appendix A. Combining Eqs. (34) and (36), this low-dimensional model system can be solved.

If two more modes \( n = 1 \) and \( n = 2 \) for wave number \( k = 2 \) are chosen, the same derivation yields the equations for \( g, a_{1,1}, a_{1,2}, a_{2,1}, \) and \( a_{2,2} \) to describe more complex physics. The resulting equations are lengthy, however, and are listed in Appendix B.

V. RESULTS AND DISCUSSIONS

A. Basic model without artificial excitation

With the numerical configuration mentioned in Sec. II, we start the simulation with excitations on a hyperbolic tangent velocity profile and then allow a transition time for as long as 10 periods of \( k = 1 \). Finally, another 10 periods are simulated and provide the data for mode decomposition. Note that the natural excitations at frequencies \( k = 1 \) and \( k = 2 \) are only used to introduce instability to the numerical simulation. So, these natural excitations are at very low amplitude and are not included in our model. They are different from the artificial excitations, which appear as extra terms in the model and are discussed in Sec. V B.
Figure 2 shows several snapshots taken from the simulation, where typical dynamics such as vortex roll-up, pairing, and merging can be clearly observed. To have a more quantitative picture of the thickness growth, we compute $d_g$ thickness as the flow develops downstream (Figure 3). It is observed that the thickness has overall growth with viscous spreading while showing events associated with vortices.

Next, the simulation data are mapped to a scaled space without downstream thickness growth. In the scaled space, we can easily get POD modes for each time-frequency. Figure 4 shows the first and second $\hat{v}$ POD modes at frequency $k = 1$. Comparing the $\hat{v}$ modes in Figure 4 to the instability mode at $k = 1$ (Figure 5) from linear instability analysis, we see the importance of forming a similar shape of the instability in POD mode $n = 2$. There was a similar observation in the modeling of TD flows.

For $k = 2$, the first two POD modes are shown in Figure 6. In a comparison to the instability mode at $k = 2$ (Figure 7), the importance of including both POD modes at this frequency is implied in a similar way.

Table I lists the energy of each mode defined by Eq. (32). The first two POD modes of each frequency contain most of the energy (total of 82% from modes (1,1) and (2,1)). However, as indicated by the comparison to the instability modes, the second POD modes are dynamically important for the instability despite their significantly reduced energy (total of 7.5% by mode (1,2) and (2,2)).

The downstream evolution of these 4 POD modes can be, therefore, obtained by a projection of direct DNS data in the scaled space as shown in Figure 8(a). As the flow develops downstream...
(along $x$), the two POD modes at $k = 2$ dominate the dynamics at first and are the most unstable modes for the initial shear thickness. In a comparison to the thickness growth, the $k = 2$ modes have clear contributions to the initial thickness growth from $x = 10$ to $x = 40$. As the shear layer thickens, the $k = 1$ modes become more unstable and quickly dominate. The appearance of $k = 1$ modes triggers a sharp increase of thickness from $x = 40$ to $x = 70$. On the other hand, $k = 2$ modes are suppressed by $k = 1$ modes through mode competition.

To obtain a 2-mode model for SD shear layers, we substitute the first and second POD modes of $k = 1$ into the coefficients and model equations defined, respectively, in Appendix A and Sec. IV, and evolve the model equations along $x$ to get the coefficients $a(x)$ and scaling function $g(x)$ as the flow develops downstream. Figure 8 compares the mode coefficients and $\delta_\alpha$ thickness calculated from the 2-mode model (Figure 8(b)) to the direction projection from DNS data (Figure 8(a)). The model captures some basic dynamics: (1) overall thickness growth; (2) oscillation frequency and amplitude of each mode for $k = 1$. Of course, the 2-mode model with single frequency cannot describe the thickness variation by the appearance of $k = 2$ vortices. The thickness variation by $k = 1$ vortices is captured but with amplified oscillation.

Similarly, to obtain a 4-mode (two-frequency) model, we substitute the first and second POD modes of each frequency $k = 1$ and $k = 2$ into the coefficients defined in Appendix B and evolve the model equations along $x$ to compute $a(x)$ and $g(x)$. The 4-mode model results are also shown and compared in Figure 8, where the mode coefficients and $\delta_\alpha$ thickness calculated from the 4-mode model (Figure 8(c)) are compared to the DNS data projection (Figure 8(a)). This time, the oscillation of POD modes from both $k = 1$ and $k = 2$ frequencies were successfully captured. The inclusion of $k = 2$ contributes to the success of presenting a small variation of thickness caused by
FIG. 6. \( \hat{v} \) for POD modes at \( k = 2 \): (a) \((k, n) = (2, 1)\) and (b) \((k, n) = (2, 2)\). The thin solid line represents the real value, the thin dashed line represents the imaginary value, and the thick solid line represents the absolute value.

FIG. 7. \( \hat{v} \) of the instability mode for \( k = 2 \). The thin solid line represents the real value, the thin dashed line represents the imaginary value, and the thick solid line represents the absolute value.

TABLE I. Energy captured by different POD modes.

<table>
<thead>
<tr>
<th>((k, n))</th>
<th>Energy (%)</th>
<th>((k, n))</th>
<th>Energy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>67.6</td>
<td>(2, 1)</td>
<td>14.4</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>5.0</td>
<td>(2, 2)</td>
<td>2.5</td>
</tr>
</tbody>
</table>

FIG. 8. Comparison of (a) direct projection from DNS data to (b) 2-mode model results and (c) 4-mode model results: ———, real part of mode coefficient \( a_{1,1} \) and \( a_{2,1} \); -- -- --, real part of mode coefficient \( a_{1,2} \) and \( a_{2,2} \); and -- -- --, shear-layer thickness \( \delta_g \).
$k = 2$ vortex roll-up, which is completely neglected by the 2-mode model. However, there is no clear improvement for the over-estimation of thickness variation by $k = 1$ vortices.

In Figure 8, the thickness growth lines show certain wavy behavior with overall viscous increase for both DNS projection and model results. Such local variation seems a result from the interaction between 2 POD modes at the same frequency. Figure 9 then gives a better understanding of such local changes by checking the phase difference between POD modes at the same frequency. It shows, for both DNS (Figure 9(a)) and models (Figures 9(b) and 9(c)), the sudden change of thickness from high to low is always accompanied by the sudden change of phase angle of 180° for the phase difference between the first and the second POD modes. This observation is true for both modes at $k = 1$ and $k = 2$. The models represent such phase angle change in the exact same way as DNS. This relation between phase angle and local variation has also been reported in TD shear layers.

To achieve a better physical understanding, Figure 10 depicts the vorticity field and shows a comparison among the original simulation, the projection of simulation data onto 2 modes which is considered an exact solution to compare to the 2-mode model, the projection of simulation data onto 4 modes which is considered an exact solution to compare to the 4-mode model, and the
results from 2-mode and 4-mode models. The 2-mode model shows the capability to describe the appearance of $k = 1$ large vortices. However, the 2-mode model produces $k = 1$ vortices from direct roll-up instead of going through the pairing and merging process of smaller vortices at $k = 2$. This results from the lack of $k = 2$ information in the 2-mode model, and it is consistent with the direct projection from DNS data onto these two modes. By adding two more modes from frequency $k = 2$, the 4-mode model has the full capability to capture the whole process including roll-up, pairing, and merging of vortices in a way consistent with the DNS projection onto these 4 modes. In fact, most characteristics of the original simulation data are represented by these 4 modes. The over-estimation of thickness variation shown before in Figure 8 now appears as vortices being over-stretched along the $y$ direction, however, without ruining the basic vortex dynamics.

B. Forced model with artificial excitations

With the external control force considered in Sec. IV, the model can be extended to study the response to artificial excitations at different frequencies. In shear layers, such frequency responses are often studied as mode competition. The competition for energy between modes at different frequencies is well-known in determining basic vortex dynamics (e.g., roll-up, pairing, and merging) in shear layers. Without external excitation, the competition between frequencies can be predicted by their instability feature; with external excitation, the forcing frequency instead determines the dominant frequency. In our study, we model the external excitation in the simplest form by assuming similar spatial functions as our base functions (i.e., POD modes) and the temporal frequencies at $k = 1$ and $k = 2$. So that, without running corresponding numerical simulation, the forced model is expected to predict similar behavior of mode competition as in real simulation.

FIG. 11. Comparison of forcing at frequency $k = 1$ with mode (1,1): different amplitudes of forcing are applied at 0 (a), 0.5 (b), 1.0 (c), and 1.5 (d); — , real part of mode coefficient $a_{1,1}$ and $a_{1,2}$; — , real part of mode coefficient $a_{1,1}$ and $a_{2,1}$; — , shear-layer thickness $\delta_y$. 
of shear layers. With the introduction of excitations at various amplitudes, the model robustness also becomes a challenge.

Since at least two frequencies need to be involved in mode competition, only the 4-mode model is considered in this section. For simplicity, only the first POD modes of each frequency are used in external forcing, although the excitation with the second POD modes can also change the vortex dynamics in a similar but less pronounced way. Figure 11 shows the results by forcing

FIG. 12. Comparison of forcing at frequency $k = 2$ with mode (2,1): different amplitudes of forcing are applied at 0 (a), 1.0 (b), 2.0 (c), and 3.0 (d); ---, real part of mode coefficient $a_{1,1}$ and $a_{2,1}$; --, real part of mode coefficient $a_{1,2}$ and $a_{2,2}$; ----, shear-layer thickness $\delta_x$.

FIG. 13. Vorticity field snapshots (top) (at $t = 624, 648, 672, 696, 720, 744, 768, 792, and 816$) and average thickness variations (bottom) from: (a) DNS projection on 4 modes, (b) 4-mode model, (c) 4-mode model with excitation at $k=1$, and (d) 4-mode model with excitation at $k=2$. Contours show vorticity $|\omega| < 0.3$. 

at \( k = 1 \) with mode (1,1). It is clearly shown that both POD modes at \( k = 1 \) are promoted and both modes at \( k = 2 \) are suppressed. The large peak of the thickness growth, which corresponds to the appearance of the \( k = 1 \) vortex, is also moved towards the upstream. On the other hand, with the excitation at \( k = 2 \) with mode (2,1) (Figure 12), the appearance of modes at \( k = 1 \) is significantly delayed. The rapid increase in the thickness is also pushed downstream and even disappears. For both figures, the excitations with larger amplitude amplify the behavior.

In Figure 13, we reconstruct the vortex field from model computation so that the characteristics of mode competition are shown in a physical and more clear manner. In Figure 13(a), snapshots from DNS projection to all 4 modes are used to show the whole evolution of vortex dynamics from roll-up, pairing, to merging. The stage for dominant frequency switching from \( k = 2 \) to \( k = 1 \) is identified by the entire vortex pairing/merging process and is boxed in Figure 13. Once vortex merging is completed, shear layer thickness reaches its peak and is noted by a vertical line. The results from the 4-mode model show similar behavior in Figure 13(b). With excitation at \( k = 1 \), Figure 13(c) shows a clearly shortened pairing/merging stage and a promotion of vortex structures at \( k = 1 \). Excitation at \( k = 2 \), Figure 13(d), on the other hand, shows a longer pairing/merging process and a delayed appearance of \( k = 1 \) vortices. Such behaviors have been observed experimentally and numerically,\(^2,16\) and here, they are qualitatively predicted by the current model without running extra simulations.

VI. CONCLUSIONS

Direct numerical simulation of a SD shear layer shows overall downstream spreading with events marked by vortex roll-up, pairing, and merging. When vortex structures translate downstream, most of these physical mechanisms remain present. However, this similarity can be damaged by the effects from mean flow variation along \( x \), and therefore, the number of required modes increases in reduced-order modeling. In order to avoid this problem, we introduce a function \( g(x) \) to scale the flow dynamically in the \( y \)-direction so that the shear-layer thickness remains the same in the scaled space. Then, a low-dimensional system evolving downstream can be built more efficiently in the new symmetry-reduced space, using traditional POD/Galerkin projection. Finally, a reconstruction equation for \( g(x) \) is derived and computed simultaneously to close the system.

The approach needs to be based on equations strictly parabolized along \( x \) and at the same time with enough terms to reproduce some key physics (i.e., instability). The parabolic requirement is satisfied by applying a thin-layer assumption on a spatially developing shear layer but without neglecting key terms for basic instability.

A 2-mode model for SD shear layers is developed to describe vortex roll-up and shear-layer thickness changes. Similar to TD cases, we need at least two POD modes for each frequency for a successful model. Each SD modes have similar shape to those of TD cases. Hence, the connection between the POD modes, and the corresponding instability modes is also observed here. The downstream development of each modes is depicted qualitatively by the 2-mode model even without enforcing mass conservation. Similarly, a 4-mode model for SD shear layers is developed to describe more complex dynamics involving the interaction between harmonics (e.g., vortex pairing and merging). There is an overall improvement for the 4-mode model, but the thickness oscillation is still over-predicted due to the lack of mass conservation. Similar differences as a result of absence/presence of mass conservation have been observed for low-dimensional TD models.\(^6,25\)

The models can describe the response to different frequencies with an extension to include effects from external excitation. The forced 4-mode model successfully predicts basic characteristics of mode competition without running the corresponding numerical simulations. The model also shows robustness under excitations at different amplitudes.

ACKNOWLEDGMENTS

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APPENDIX A: EQUATIONS AND COEFFICIENTS FOR THE 2-MODE MODEL

1. Projection of left-hand-side

\[
\left\langle \tilde{q}, \frac{\partial \Phi_1}{\partial \hat{t}} \right\rangle = T \left[ (u_0 \tilde{u}_{1,1} + u_0 \hat{v}_{1,1}) \right] \hat{\xi}_{1,1} + T \left[ (u_0 \tilde{u}_{1,2} + u_0 \hat{v}_{1,2}) \right] \hat{\xi}_{1,2}
\]

\[= T(b_{11} \hat{\xi}_{1,1} + b_{12} \hat{\xi}_{1,2}), \quad (A1)\]

so that coefficients are defined

\[b_{11} = \int (u_0 \tilde{u}_{1,1} + u_0 \hat{v}_{1,1}) dy, \quad (A2)\]

\[b_{12} = \int (u_0 \tilde{u}_{1,2} + u_0 \hat{v}_{1,2}) dy. \quad (A3)\]

Similarly,

\[b_{21} = \int (u_0 \tilde{u}_{1,2} + u_0 \hat{v}_{1,2}) dy, \quad (A4)\]

\[b_{22} = \int (u_0 \tilde{u}_{1,2} + u_0 \hat{v}_{1,2}) dy. \quad (A5)\]

2. Projection of nonlinear terms

\[
\left\langle N_g(q, \tilde{q}), \Phi_1 \right\rangle = -Tg \left[ \left( \tilde{\xi}_0 \frac{d\tilde{u}_{1,1}}{dy} \tilde{u}_{1,1} + \tilde{\xi}_0 \frac{d\tilde{v}_{1,1}}{dy} \tilde{v}_{1,1} \right) \right] a_{1,1}
\]

\[-Tg \left[ \left( \tilde{\xi}_0 \frac{d\tilde{u}_{1,2}}{dy} \tilde{u}_{1,2} + \tilde{\xi}_0 \frac{d\tilde{v}_{1,2}}{dy} \tilde{v}_{1,2} \right) \right] a_{1,2}
\]

\[-Tg \left[ \left( \tilde{\xi}_{1,1} \frac{d\hat{u}_{1,1}}{dy} \hat{u}_{1,1} + \tilde{\xi}_{1,1} \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1} \right) \right] a_{1,1}
\]

\[-Tg \left[ \left( \tilde{\xi}_{1,2} \frac{d\hat{u}_{1,2}}{dy} \hat{u}_{1,2} + \tilde{\xi}_{1,2} \frac{d\hat{v}_{1,2}}{dy} \hat{v}_{1,2} \right) \right] a_{1,2}
\]

\[-Tg \left[ \left( \tilde{\xi}_{1,1} \frac{d\hat{u}_{1,1}}{dy} \hat{u}_{1,1} \right) \right] a_{1,1} - Tg \left[ \left( \tilde{\xi}_{1,2} \frac{d\hat{u}_{1,2}}{dy} \hat{u}_{1,2} \right) \right] a_{1,2}, \quad (A6)\]

where \(\tilde{\xi}_0 = 0\) is applied. So we have

\[c_{11} = -\int \left( \tilde{\xi}_{1,1} \frac{d\hat{u}_{1,1}}{dy} \right) dy, \quad (A7)\]

\[c_{12} = -\int \left( \tilde{\xi}_{1,2} \frac{d\hat{u}_{1,1}}{dy} \right) dy. \quad (A8)\]

Similarly,

\[c_{21} = -\int \left( \tilde{\xi}_{1,1} \frac{d\hat{u}_{1,2}}{dy} \right) dy, \quad (A9)\]

\[c_{22} = -\int \left( \tilde{\xi}_{1,2} \frac{d\hat{u}_{1,2}}{dy} \right) dy. \quad (A10)\]
3. Projection of linear terms

\[
\langle L_\delta(q), \Phi_{1,1} \rangle = -T \left( \frac{2\pi i}{T} \right) \left[ \left( \hat{u}_{1,1} \hat{u}_{1,1}^* + \hat{v}_{1,1} \hat{v}_{1,1}^* \right) dy \right] a_{1,1} \\
+ \frac{1}{Re} g^2 T \left\{ \left[ \left( \frac{d^2 \hat{u}_{1,1}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,1}^* \right) dy \right] a_{1,1} \\
+ \left[ \left( \frac{d^2 \hat{u}_{1,1}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,1}^* \right) dy \right] a_{1,2} \right\} \\
= T(c_{13} a_{1,1} + \frac{g^2}{Re} [d_{11} a_{1,1} + d_{12} a_{1,2}]), \tag{A11}
\]

where

\[
c_{13} = - \left( \frac{2\pi i}{T} \right) \left( \hat{u}_{1,1} \hat{u}_{1,1}^* + \hat{v}_{1,1} \hat{v}_{1,1}^* \right) dy, \tag{A12}
\]

\[
d_{11} = \left[ \left( \frac{d^2 \hat{u}_{1,1}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,1}^* \right) dy \right], \tag{A13}
\]

\[
d_{12} = \left[ \left( \frac{d^2 \hat{u}_{1,1}}{dy^2} \hat{u}_{1,1}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,1}^* \right) dy \right]. \tag{A14}
\]

Similarly

\[
c_{23} = - \left( \frac{2\pi i}{T} \right) \left( \hat{u}_{1,2} \hat{u}_{1,2}^* + \hat{v}_{1,2} \hat{v}_{1,2}^* \right) dy, \tag{A15}
\]

\[
d_{21} = \left[ \left( \frac{d^2 \hat{u}_{1,1}}{dy^2} \hat{u}_{1,2}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,2}^* \right) dy \right], \tag{A16}
\]

\[
d_{22} = \left[ \left( \frac{d^2 \hat{u}_{1,1}}{dy^2} \hat{u}_{1,2}^* + \frac{d^2 \hat{v}_{1,1}}{dy^2} \hat{v}_{1,2}^* \right) dy \right]. \tag{A17}
\]

4. Projection of thickness correction terms

The correction term of downstream evolution resulted by the scaling is

\[
\langle -\hat{A} \frac{g^2}{g} \frac{d\hat{q}}{dy}, \Phi_{1,1} \rangle = -T \frac{g}{g} \left\{ \left[ \left( \frac{d\hat{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^* \right) dy \right] a_{1,1} \\
- T \frac{g}{g} \left[ \left( \frac{d\hat{u}_{1,2}}{dy} \hat{u}_{1,2}^* + \frac{d\hat{v}_{1,2}}{dy} \hat{v}_{1,2}^* \right) dy \right] a_{1,2} \\
- T \frac{g}{g} \left[ \left( \frac{d\hat{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \frac{d\hat{v}_{1,2}}{dy} \hat{v}_{1,2}^* \right) dy \right] a_{1,1} \\
- T \frac{g}{g} \left[ \left( \frac{d\hat{u}_{1,2}}{dy} \hat{u}_{1,2}^* + \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^* \right) dy \right] a_{1,2} \right\} \\
= T \frac{g}{g} (e_{11} a_{1,1} + e_{12} a_{1,2}). \tag{A18}
\]

Coefficients are

\[
e_{11} = - \left[ \left( \frac{d\hat{u}_{1,1}}{dy} \hat{u}_{1,1}^* + \frac{d\hat{v}_{1,1}}{dy} \hat{v}_{1,1}^* \right) dy \right], \tag{A19}
\]

\[
e_{12} = - \left[ \left( \frac{d\hat{u}_{1,2}}{dy} \hat{u}_{1,2}^* + \frac{d\hat{v}_{1,2}}{dy} \hat{v}_{1,2}^* \right) dy \right]. \tag{A20}
\]
where \( \dot{v}_0 = 0 \) is applied again. Similarly,

\[
e_{21} = - \int \left( u_0 \frac{d\dot{u}_{1,1}}{dy} \ddot{u}_{1,1} + u_0 \frac{d\dot{v}_{1,1}}{dy} \ddot{v}_{1,1} + \dot{u}_0 \frac{du_{0,1}}{dy} \ddot{u}_{0,1} \right) dy,
\]

\[
e_{22} = - \int \left( u_0 \frac{d\dot{u}_{1,2}}{dy} \ddot{u}_{1,2} + u_0 \frac{d\dot{v}_{1,2}}{dy} \ddot{v}_{1,2} + \dot{u}_0 \frac{du_{0,2}}{dy} \ddot{u}_{0,2} \right) dy.
\]  

\[\text{(A21)}\]

\[\text{(A22)}\]

5. Projection of external forcing terms

\[
\langle f_q, \Phi_{1,1} \rangle = T \left[ \int (A_{1,1} \ddot{u}_{1,1} + A_{1,1} \ddot{v}_{1,1}) dy \right] + T \left[ \int (A_{1,2} \ddot{u}_{1,2} + A_{1,2} \ddot{v}_{1,2}) dy \right] = T(A_{1,1}).
\]

\[\text{(A23)}\]

Similarly

\[
\langle f_q, \Phi_{1,2} \rangle = T(A_{1,2}).
\]

\[\text{(A24)}\]

6. Terms for thickness evolution

\[
\left\langle yu \frac{\partial \tilde{u}}{\partial y}, yu_0 \frac{\partial u_0}{\partial y} \right\rangle = T \left( \left( yu_0 \frac{du_0}{dy} \right)^2 dy + T \left[ \int \left( \ddot{u}_{1,1} + \ddot{v}_{1,1} \right) \frac{du_0}{dy} dy \right] a_{1,1} \ddot{a}_{1,1} \right.
\]

\[
+ T \left[ \int \left( \ddot{u}_{1,2} + \ddot{v}_{1,2} \right) \frac{du_0}{dy} dy \right] a_{1,2} \ddot{a}_{1,2} \right)
\]

\[
+ T \left[ \int \left( \ddot{u}_{1,1} + \ddot{v}_{1,1} \right) \frac{du_0}{dy} dy \right] a_{1,1} \ddot{a}_{1,2} \right)
\]

\[
+ T \left[ \int \left( \ddot{u}_{1,2} + \ddot{v}_{1,2} \right) \frac{du_0}{dy} dy \right] a_{1,2} \ddot{a}_{1,2} \right)
\]

\[\text{(A25)}\]

and

\[
\left\langle f_g^1, yu_0 \frac{\partial u_0}{\partial y} \right\rangle = \left\langle N_g^1(\tilde{q}, \tilde{q}), yu_0 \frac{\partial u_0}{\partial y} \right\rangle + \left\langle L_g^1(\tilde{q}, \tilde{q}), yu_0 \frac{\partial u_0}{\partial y} \right\rangle
\]

\[
- T_g \left[ \int \left( \ddot{u}_{1,1} + \ddot{v}_{1,1} \right) yu_0 \frac{du_0}{dy} dy \right] a_{1,1} \ddot{a}_{1,1} \right)
\]

\[
- T_g \left[ \int \left( \ddot{u}_{1,2} + \ddot{v}_{1,2} \right) yu_0 \frac{du_0}{dy} dy \right] a_{1,2} \ddot{a}_{1,2} \right)
\]

\[
- T_g \left[ \int \left( \ddot{u}_{1,1} + \ddot{v}_{1,1} \right) yu_0 \frac{du_0}{dy} dy \right] a_{1,1} \ddot{a}_{1,2} \right)
\]

\[
- T_g \left[ \int \left( \ddot{u}_{1,2} + \ddot{v}_{1,2} \right) yu_0 \frac{du_0}{dy} dy \right] a_{1,2} \ddot{a}_{1,2} \right)
\]

\[\text{and} \frac{1}{Re} \ddot{a}^2 \int \left( \frac{d^2 u_0}{dy^2} yu_0 \frac{du_0}{dy} \right) dy.
\]

\[\text{(A26)}\]

Therefore, the coefficients in thickness evolution equation (36) are

\[
b_0 = \int \left( yu_0 \frac{du_0}{dy} \right)^2 dy,
\]

\[\text{(A27)}\]
and the differential equations for scaling variable $g$ and coefficient vector are

$$
\mathbf{a} = (a_{11} \ a_{12})^T,
$$

$$
\dot{g} = \frac{(c_{01}a_{11}a_{11} + c_{02}a_{11}a_{12} + c_{03}a_{12}a_{11} + c_{04}a_{12}a_{12}^*) g^2 + \frac{1}{Re} d_0 g^3}{b_0 + b_{01}a_{11}a_{11} + b_{02}a_{11}a_{12} + b_{03}a_{12}a_{11} + b_{04}a_{12}a_{12}^*},
$$

and

$$
\mathbf{B}_a = \left( g \mathbf{C} + \frac{1}{Re} g^2 \mathbf{D} + \frac{\dot{g}}{g} \mathbf{E} \right) \mathbf{a} + \mathbf{F},
$$

where the matrices $\mathbf{B}$, $\mathbf{C}$, $\mathbf{A}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$ for the two-mode case were defined previously in Sec. IV D.

**APPENDIX B: EQUATIONS FOR THE 4-MODE MODEL**

The differential equations for scaling variable $g$ and coefficient vector

$$
\mathbf{a} = (a_{11} \ a_{12} \ a_{21} \ a_{22})^T
$$

for the 4-mode model are given as

$$
\dot{g} = \frac{C_0}{B_0} g^2 + \frac{1}{Re B_0} g^3,
$$

where coefficients $B_0$ and $C_0$ are defined by

$$
B_0 = b_0 + b_{01}a_{11}a_{11} + b_{02}a_{11}a_{12} + b_{03}a_{12}a_{11} + b_{04}a_{12}a_{12}^* + b_{05}a_{21}a_{21} + b_{06}a_{21}a_{22}^* + b_{07}a_{22}a_{21} + b_{08}a_{22}a_{22}^*,
$$

and

$$
C_0 = c_{01}a_{11}a_{11} + c_{02}a_{11}a_{12} + c_{03}a_{12}a_{11} + c_{04}a_{12}a_{12}^* + c_{05}a_{21}a_{21} + c_{06}a_{21}a_{22}^* + c_{07}a_{22}a_{21} + c_{08}a_{22}a_{22}^*.
$$
and

$$B\hat{a} = \left( C + \frac{g^2}{Re} D + \frac{g}{\mu} E \right) a + N_1 + \frac{g}{\mu} N_2 + F,$$  \hspace{1cm} \text{(B4)}$$

where matrices $N_1$ and $N_2$ include all terms nonlinear to $a_{k,n}$ as

$$N_1 = \left[ \begin{array}{c}
g(c_{112}a_{1,1}^2a_{2,1} + c_{1122}a_{1,1}^1a_{2,1} + c_{1221}a_{1,2}^2a_{2,1} + c_{1222}a_{1,2}^1a_{2,2}) \\
g(c_{132}a_{1,1}^1a_{2,1} + c_{1322}a_{1,1}^1a_{2,2} + c_{1421}a_{1,2}^1a_{2,1} + c_{1422}a_{1,2}^1a_{2,2}) \\
g(c_{151}a_{1,1}^1a_{1,1} + c_{1512}a_{1,1}^1a_{1,2} + c_{1522}a_{1,2}a_{1,1}) \\
g(c_{161}a_{1,1}^1a_{1,1} + c_{162}a_{1,1}a_{1,2} + c_{1622}a_{1,2}a_{1,2}) \end{array} \right],$$  \hspace{1cm} \text{(B5)}$$

and

$$N_2 = \left[ \begin{array}{c}
e_{112}a_{1,1}^1a_{2,1} + e_{1122}a_{1,1}^1a_{2,1} + e_{1221}a_{1,2}^2a_{2,1} + e_{1222}a_{1,2}^1a_{2,2} \\
e_{132}a_{1,1}^1a_{2,1} + e_{1322}a_{1,1}^1a_{2,2} + e_{1421}a_{1,2}^1a_{2,1} + e_{1422}a_{1,2}^1a_{2,2} \\
e_{151}a_{1,1}a_{1,1} + e_{1512}a_{1,1}a_{1,2} + e_{1522}a_{1,2}a_{1,2} \\
e_{161}a_{1,1}a_{1,1} + e_{162}a_{1,1}a_{1,2} + e_{1622}a_{1,2}a_{1,2} \end{array} \right],$$  \hspace{1cm} \text{(B6)}$$

and other matrices for terms linear to $a_{k,n}$ have blocks of zeroes as shown below,

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$  \hspace{1cm} \text{(B7)}$$

with sub-matrices defined by

$$B_1 = \begin{bmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{121} & b_{122} \\ b_{221} & b_{222} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} gc_{111} + c_{111} & gc_{112} \\ gc_{211} & gc_{212} + c_{312} \end{bmatrix}, \quad C_2 = \begin{bmatrix} gc_{121} + c_{321} & gc_{122} \\ gc_{221} & gc_{222} + c_{322} \end{bmatrix},$$

$$D_1 = \begin{bmatrix} d_{111} & d_{112} \\ d_{211} & d_{212} \end{bmatrix}, \quad D_2 = \begin{bmatrix} d_{121} & d_{122} \\ d_{221} & d_{222} \end{bmatrix},$$

$$E_1 = \begin{bmatrix} e_{111} & e_{112} \\ e_{211} & e_{212} \end{bmatrix}, \quad E_2 = \begin{bmatrix} e_{121} & e_{122} \\ e_{221} & e_{222} \end{bmatrix}.$$  

The equations for each coefficients are too many to be listed here.