Proceedings of ITP-07
2007 Interdisciplinary Transport Phenomena V
Bansko, Bulgaria, October 14-19, 2007

ITP-07-26

HAMILTONIAN STRUCTURE AND DYNAMICS OF A NEUTRALLY BUOYANT RIGID SPHERE INTERACTING WITH THIN VORTEX RINGS

Banavara N. Shashikanth, *
Department of Mechanical Engineering,
New Mexico State University,
Las Cruces, New Mexico 88003, USA
Email: shashi@nmsu.edu

Artan Sheshmani,
Department of Mathematics,
University of Illinois at Urbana-Champaign,
Urbana, Illinois 61801, USA

Scott Kelly,
Department of Mechanical Science and Engineering,
University of Illinois at Urbana-Champaign,
Urbana, Illinois 61801, USA

Mingjun Wei,
Department of Mechanical Engineering,
New Mexico State University,
Las Cruces, New Mexico 88003, USA

ABSTRACT

In a previous paper, we presented a (noncanonical) Hamiltonian model for the dynamic interaction of a neutrally buoyant rigid body of arbitrary smooth shape with N closed vortex filaments of arbitrary smooth shape, modeled as curves, in an infinite ideal fluid in $\mathbb{R}^3$. In this paper, we focus on the case when the body is a smooth sphere. The equations of motion and Hamiltonian structure of this dynamic system, which follow from the general model, are presented. Following this, we impose the constraint of axisymmetry on the entire system and look at the case when the rings are all circles perpendicular to a common axis of symmetry passing through the center of the sphere. This axisymmetric model, in our idealized framework, is governed by ordinary differential equations and is, relatively speaking, easily integrated numerically. Finally, we present some plots of dynamic orbits of the axisymmetric system.

NOMENCLATURE

$\mathbf{a}$: radius vector of sphere
$a = |\mathbf{a}|$
Γ_i: strength of ith ring
$\mathbf{U}$: translation velocity vector of sphere$= U\mathbf{b}$ in the axisymmetric model
$\mathbf{L}$: 'linear momentum' of body+fluid system$= L\mathbf{b}$ in the axisymmetric model
M: mass plus added mass of sphere
C_i: arc-length parameterized curve representing ith ring
s_i: arc-length parameter of ith ring
R_i: radius of ith ring in the axisymmetric model
z_i: position of (center of) ith ring along axis of symmetry measured from origin of body-fixed frame
$\mathbf{u}_{R,i}$: velocity field due to the ith ring in unbounded flow
$\mathbf{u}_{I,i}$: velocity field of image of ith ring
$\mathbf{u}_{SI,i}$: self-induced velocity of ith ring, for the axisymmetric model $\mathbf{u}_{SI,i} = u_{LI,i}\mathbf{b}$ where LI stands for the local induction approximation
$\Phi_{I,i}$: velocity potential of $\mathbf{u}_{I,i}$
Φ_B: velocity potential of the Kirchhoff velocity field
$\nabla\Phi_B$: Kirchhoff velocity field
$\mathbf{n}_i$: principal unit normal field on ith ring

*Address all correspondence to this author.

b: unit binormal field—parallel to axis of symmetry for all rings

$\mathbf{t}_i$: unit tangent field on ith ring

N_{ij}: $\mathbf{n}_i$ component of $\mathbf{u}_{R,j}{}_{|_{C_i}}$

B_{ij}: $\mathbf{b}$ component of $\mathbf{u}_{R,j}{}_{|_{C_i}}$

$\mathcal{N}_{ij}$: $\mathbf{n}_i$ component of $\mathbf{u}_{I,j}{}_{|_{C_i}}$

$\mathcal{B}_{ij}$: $\mathbf{b}$ component of $\mathbf{u}_{I,j}{}_{|_{C_i}}$

$\mathfrak{N}_i$: $\mathbf{n}_i$ component of $\nabla\Phi_{B}{}_{|_{C_i}}$

$\mathfrak{B}_i$: $\mathbf{b}$ component of $\nabla\Phi_{B}{}_{|_{C_i}}$

∂B: surface of sphere

INTRODUCTION

The objective of this paper is essentially twofold. First, we want to present the equations of motion and the Hamiltonian structure of the system consisting of a neutrally buoyant rigid sphere interacting dynamically with N arbitrarily-shaped and arbitrarily-oriented vortex rings, modeled as N closed curves in $\mathbb{R}^3$, as in Figure 1. These equations and Hamiltonian structure will be derived as a special case of the model described in [1]. The simple geometry of the sphere allows an explicit representation of the image velocity field of the rings and we will follow the work of [2] for this. Second, with a view to studying dynamic orbits of such a system, we focus on the case of an axisymmetric configuration in which the rings are all circles (in parallel planes) with centers along a common line passing through the center of the sphere, as in Figure 2. For this axisymmetric case, the system equations become ordinary differential equations and are thus easily integrated numerically.

Motivations for constructing models like these come from locomotion problems in a fluid environment–both in nature and engineering–such as, for example, the swimming of neutrally buoyant fish and the energy-efficient design of small, autonomous underwater vehicles where coherent vortical structures in the vicinity of the moving body play an important role. Other potential applications, more in line with the theme of this conference, include transport phenomena of small particles in a fluid environment or of particles in microgravity environments. The models incorporate nonlinear effects and, within an inviscid framework, fully couple the solid-fluid dynamics. It is of course necessary to develop these models more, in particular to include the effects of fluid viscosity and, perhaps, turbulence. Nevertheless, as a start, these non-trivial, low-dimensional models provide a platform for applying the powerful theoretical tools of dynamical systems and nonlinear control to study the complex phenomena of solid-fluid interactions in locomotion problems.

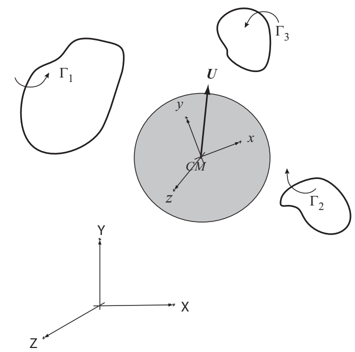

Figure 1. A SCHEMATIC VIEW OF N RINGS OF ARBITRARY SHAPE AND A NEUTRALLY BUOYANT RIGID SPHERE INTERACTING WITH EACH OTHER IN AN INVISCID, INCOMPRESSIBLE FLOW.

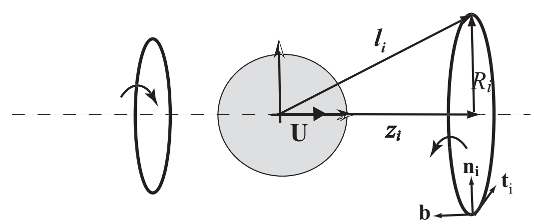

Figure 2. A SCHEMATIC VIEW OF N AXISYMMETRIC CIRCULAR RINGS AND A NEUTRALLY BUOYANT RIGID SPHERE INTERACTING WITH EACH OTHER IN AN INVISCID, INCOMPRESSIBLE FLOW.

EQUATIONS OF MOTION

Problem Setting and Assumptions.

The problem setting and underlying assumptions are the same as in [1], however, for the sake of completeness, we describe these again. We consider a rigid smooth sphere that is immersed in an ideal (inviscid, incompressible) fluid. The fluid extends to infinity in all directions away from the sphere. Both the sphere and fluid have uniform density and, moreover, the sphere is assumed to be neutrally buoyant, i.e., its density is equal to the density of the fluid which, without loss of generality, can be taken to be equal to unity. The boundary conditions are the matching of the normal velocity ("free-slip") conditions on the sphere's surface and the fluid at rest at infinity. The vorticity field of the fluid is compact and assumed to be a delta distribution supported on N vortex rings of arbitrary shape, non-intersecting with each other and the sphere. As stated before, the N rings can be viewed as N arbitrary smooth closed curves in $\mathbb{R}^3$.

The sphere is free to move under the instantaneous pressure field induced on its surface by the fluid. The motion of the sphere in turn induces a motion of the fluid. The velocity field of the fluid consists, as per the Hodge decomposition [3], of two parts: (i) the irrotational, Kirchhoff potential field which satisfies the normal velocity matching condition on the sphere surface and (ii) the divergence-free, rotational field due to the N vortex rings in the presence of the sphere with zero normal velocity on the sphere surface. This second part further decomposes into two components, namely the velocity field due to the rings in the absence of the sphere and the velocity field due to the image vorticity inside the sphere which enforces the zero normal velocity condition on the sphere surface.

The Lie-Poisson Equations of the Rings–Sphere System.

The Lie-Poisson, or the momentum, equations [3] for the rings-sphere system are obtained from the Lie-Poisson equations in [1] by making the following observation. For the special geometry of the sphere and for the inviscid, free-slip boundary conditions in this problem, the angular velocity of the sphere Ω cannot affect and cannot be affected by the dynamics of the system. Hence, without loss of generality, set $\Omega = 0$. Consequently, note that the transformation to the body-fixed frame is just a translation and involves no rotation of the frame.

Following these steps the Lie-Poisson equations of the system are obtained simply as:

$$\frac{d\mathbf{L}}{dt} = 0, \tag{1}$$

where

$$\mathbf{L} = M\mathbf{U} + \mathbf{P} \tag{2}$$

with M equal to the mass plus added mass of the sphere, $\mathbf{U}$ the sphere center-of-mass velocity and

$$\mathbf{P} = \frac{1}{2}\sum_{i=1}^{N}\Gamma_i\left(\oint_{C_i}(\mathbf{l}_i(s_i)\times\mathbf{t}_i(s_i))ds_i\right) + \frac{1}{2}\int_{\partial B}\mathbf{l}\times(\mathbf{n}\times\mathbf{u}_V)dA. \tag{3}$$

In the above, Γ_i is the strength of the ith ring, $\mathbf{l}$ denotes the position vector of points in the body-fixed frame with $l = \|\mathbf{l}\|$, $\mathbf{l}_i$ is $\mathbf{l}$ for points on the ith ring with $l_i = \|\mathbf{l}_i\|$, C_i is the parameterized curve denoting the ith ring in the body-fixed frame, $\mathbf{t}_i$ is the unit tangent vector field on the ith ring, s_i is the arc-length parameter for the ith ring and ∂B denotes the surface of the sphere. The term $\mathbf{u}_V$ is defined a little later.

Note that for the simple geometry of the sphere the term $\mathbf{P}$ can be written as

$$\mathbf{P} = \frac{1}{2}\sum_{i=1}^{N}\Gamma_i\left(\oint_{C_i}(\mathbf{l}_i(s_i)\times\mathbf{t}_i(s_i))ds_i\right) - \frac{a}{2}\int_{\partial B}\mathbf{u}_V dA, \tag{4}$$

using the boundary condition (8) for $\mathbf{u}_V$ and where a is the sphere radius. As in [1], our convention for unit normals is that they point inwards.

Evolution Equations for the Rings

The evolution equations for the rings are obtained by applying a fundamental law of inviscid vortex motion, namely that singular distributions of vorticity (for example rings, point vortices, and vortex sheets), are convected by the fluid flow, see [4] (Ch.1 and 2) for more details. In the body-fixed frame, these equations are

$$\frac{\partial C_i}{\partial t} + \mathbf{U}^n_{|C_i} = \left(\sum_{j,j\neq i}^{N}\mathbf{u}_{V,j} + \nabla\Phi_B + \mathbf{u}_{I,i}\right)^n_{|C_i} + (\mathbf{u}_{SI,i})^n, \quad i = 1,\cdots,N \tag{5}$$

where $\mathbf{u}_V$ and $\nabla\Phi_B$ are the Hodge components of the total fluid velocity field $\mathbf{u}$,

$$\mathbf{u} = \nabla\Phi_B + \mathbf{u}_V, \tag{6}$$

satisfying

$$\nabla^2\Phi_B = 0 \quad \text{and} \quad \nabla\cdot\mathbf{u}_V = 0, \tag{7}$$

in the infinite fluid domain D and the boundary conditions

$$\nabla\Phi_B\cdot\mathbf{n} = \mathbf{U}\cdot\mathbf{n} \quad \text{and} \quad \mathbf{u}_V\cdot\mathbf{n} = 0, \tag{8}$$

on the body surface ∂B. Moreover, at infinity $\nabla\Phi_B, \mathbf{u}_V \to 0$. The term $\mathbf{u}_{V,j_{|C_i}}$ in (5) is just $\mathbf{u}_V$ due to the jth ring at the location of the ith ring and is given by

$$\mathbf{u}_{V,j_{|C_i}} = \frac{\Gamma_j}{4\pi}\oint_{C_j}\frac{\mathbf{t}_j\times(\mathbf{l}_i-\mathbf{l}_j)\,ds_j}{\left|\mathbf{l}_i-\mathbf{l}_j\right|^3} + \mathbf{u}_{I,j_{|C_i}} \equiv \mathbf{u}_{R,j_{|C_i}} + \mathbf{u}_{I,j_{|C_i}}, \tag{9}$$

where $\mathbf{u}_{R,j}$ is the Biot-Savart component and $\mathbf{u}_{I,j}$ is the velocity field of the image vorticity associated with the jth ring and is obtained as the solution to the following (Neumann) problem

$$\nabla \times \mathbf{u}_{I,j} = 0 \text{ in } D, \tag{10}$$

$$\mathbf{u}_{R,j} \cdot \mathbf{n} = -\mathbf{u}_{I,j} \cdot \mathbf{n} \text{ on } \partial B. \tag{11}$$

The term $\mathbf{u}_{SI,i}$ is the self-induced velocity field of the ith ring on the ith ring appropriately regularized. The superscript n in (5) simply means that only the non-parallel components of the vector field contribute to changes in the curve shape. More precisely, for any vector field $\mathbf{X}$, define its non-parallel components on C_i by

$$(\mathbf{X})^n = \mathbf{t}_i \times (\mathbf{X} \times \mathbf{t}_i).$$

Finally, we note that Φ_B is the potential function of the Kirchhoff field induced by the motion of the body which has a linear decomposition in terms of the components of the velocity $\mathbf{U}$ of the body center (of mass) (see [5]):

$$\Phi_B = \phi_x u + \phi_y v + \phi_z w, \tag{12}$$

where $\mathbf{U} \equiv (u, v, w)$ and the functions ϕ_x, ϕ_y and ϕ_z are given by

$$\phi_x(\mathbf{l}) = -\frac{a^3 x}{2\,|\,\mathbf{l}\,|^3}, \quad \phi_y(\mathbf{l}) = -\frac{a^3 y}{2\,|\,\mathbf{l}\,|^3}, \quad \phi_z(\mathbf{l}) = -\frac{a^3 z}{2\,|\,\mathbf{l}\,|^3},$$

Thus, the final coupled equations of motion of the dynamically interacting system of a rigid smooth sphere and N vortex rings of arbitrary shape, in a body-fixed frame, are given by (1) and (5):

$$\frac{d\mathbf{L}}{dt} = 0 \tag{13}$$

$$\frac{\partial C_i}{\partial t} + \mathbf{U}^n_{|C_i} = \left(\sum_{j,j\neq i}^{N} \mathbf{u}_{V,j} + \nabla\Phi_B + \mathbf{u}_{I,i} \right)^n_{|C_i} + (\mathbf{u}_{SI,i})^n, \quad i = 1, \cdots, N \tag{14}$$

Note that the equation for the angular momentum (impulse) A of the system (see [1]) is decoupled from the above equations and hence plays no role in the dynamics of the system.

Hamiltonian Structure of the Equations

In this section it will be shown that (13) and (14), have a Hamiltonian structure relative to the kinetic energy Hamiltonian of the system and an appropriate Poisson bracket.

Phase Space of the System.

Before defining the Hamiltonian function we need to define the system phase space. The system phase space is

$$P := P_b \times P_R \equiv \mathbb{R}^{3*} \times (S \backslash \Delta), \tag{15}$$

where P_b is the reduced phase space of the sphere in the body-fixed frame and is identified with $\mathbb{R}^{3*}$, the dual of $\mathbb{R}^3$, which is in turn identified with $\mathbb{R}^3$ using the standard Euclidean pairing. P_R is the phase space of the rings in the body-fixed frame identified with S the space of N (smooth) closed curves in $\mathbb{R}^3 \backslash B$ minus Δ, the intersection set of the curves. Note that the phase space excludes all intersections of the rings amongst themselves and with the body.

The Kinetic Energy.

Consider the kinetic energy of the rings plus sphere system:

$$K = \frac{1}{2} \int_D \langle \mathbf{u}, \mathbf{u} \rangle \, dV + \frac{1}{2} \langle \mathbf{U}, M_b \mathbf{U} \rangle,$$

where M_b is the mass of the sphere. Using the Hodge and the Kirchhoff decompositions, this can be written as

$$K = \frac{1}{2} \int_D \langle \mathbf{u}_V, \mathbf{u}_V \rangle + \frac{1}{2} \langle \mathbf{U}, M\mathbf{U} \rangle.$$

As noted previously and discussed in detail in [1], $\mathbf{u}_V$ has singularities in the self-induced velocity and this results in the divergence of K.

The Hamiltonian function of the system, $H : P_R \to \mathbb{R}$, is the kinetic energy K written in terms of the phase space variables and with the self-induced kinetic energy term appropriately regularized:

$$H(\mathfrak{s}, \mathbf{L}) = \frac{1}{2} \sum_{i=1}^{N} \int_D \langle \mathbf{u}_{V,i}, \sum_{j\neq i}^{N} \mathbf{u}_{V,j} \rangle dV + \frac{1}{2} \sum_{i=1}^{N} \int_D \langle \mathbf{u}_{R,i}, \mathbf{u}_{I,i} \rangle dV + H_{SI} + \frac{1}{2} \left\langle \mathbf{L} - \mathbf{P}, M^{-1}(\mathbf{L} - \mathbf{P}) \right\rangle, \tag{16}$$

where $\mathfrak{s} \equiv (C_1, \ldots\ldots, C_N) \in S$ and H_{SI} is the regularized self-induced kinetic energy term. In the above, we have used the fact that $\int_D \langle \mathbf{u}_{V,i}, \mathbf{u}_{I,i} \rangle dV = 0$. Note that $\mathbf{P}$ as defined by equation (3) is also a function on S.

Poisson Brackets.

Consider the following Poisson bracket on $P \equiv P_b \times P_R = \mathbb{R}^{3*} \times S \backslash \Delta$. For $F, G : P \to \mathbb{R}$,

$$\{F,G\}_P = \left\{F|_{P_b}, G|_{P_b}\right\} + \{F|_{P_v}, G|_{P_v}\}, \tag{17}$$

where the first bracket is the trivial bracket and is what the Lie-Poisson bracket on $\mathfrak{se}(3)^*$, see [3], reduces to in the absence of the rotational subgroup.

$$\left\{\tilde{F}, \tilde{G}\right\}(\mathbf{L}) = 0 \tag{18}$$

for all $\tilde{F}, \tilde{G} : \mathbb{R}^{3*} \longrightarrow \mathbb{R}$ and $\mathbf{L} \in \mathbb{R}^{3*}$.

The second bracket is the canonical bracket associated with the symplectic form on the phase space of rings/filaments in $\mathbb{R}^3$ derived by Marsden and Weinstein [6]. As shown in [1], this is given by

$$\begin{aligned}\left\{\hat{F}, \hat{G}\right\}(\mathfrak{s}) = \sum_{i=1}^{N} \Gamma_i \oint_i \Bigg(\left\langle \frac{\delta \hat{F}}{\delta C_i}, \mathbf{n}_i \right\rangle \left\langle \frac{\delta \hat{G}}{\delta C_i}, \mathbf{b}_i \right\rangle \\ - \left\langle \frac{\delta \hat{F}}{\delta C_i}, \mathbf{b}_i \right\rangle \left\langle \frac{\delta \hat{G}}{\delta C_i}, \mathbf{n}_i \right\rangle \Bigg) ds_i, \end{aligned} \tag{19}$$

$$\equiv \sum_{i=1}^{N} \Gamma_i \oint_i \left\langle \frac{\delta \hat{F}}{\delta C_i} \times \frac{\delta \hat{G}}{\delta C_i}, \mathbf{t}_i \right\rangle ds_i$$

for functions $\hat{F}, \hat{G} : S \longrightarrow \mathbb{R}$. In the above, $\mathbf{n}_i, \mathbf{t}_i$ and $\mathbf{b}_i$ are the unit principal normal, unit tangent and unit binormal vector fields, respectively, along the ith ring, and the functional derivatives are also identified with vector fields along the rings.

Thus, finally, we have:

Lemma. *Equations* (13) *and* (14) *form a Hamiltonian system for the Hamiltonian function H, given by (16), on the Poisson manifold P, defined by (15), equipped with the Poisson bracket (17).*

Proof. The proof follows from the proof for general smooth body shapes, as given in [1], by applying $\Omega = 0$ and H independent of A everywhere.

Remarks. Note that as in [1] we are again assuming that a consistency condition relative to the Hamiltonian structure holds for the regularized self-induction terms. In other words, the self-induced velocity field satisfies $\mathbf{u}^n_{SI} = \frac{\delta H_{SI}}{\delta C_i} \times \mathbf{t}_i$ everywhere on the ith ring.

To proceed further with equations (13) and (14) and study the dynamics, it is useful to first write down expressions for $\mathbf{u}_{I,i}$ (see (9)) which the simple geometry of the sphere allows. It is also necessary to come up with a choice of $\mathbf{u}_{SI,i}$ which we discuss later.

Velocity Field of the Image Vorticity.

For the structure of $\mathbf{u}_I$, the potential field outside the sphere due to the image vorticity, we follow the technique suggested by Knio and Ting [2]. This work, drawing from the work of Weiss [8] on the image of a potential field in a sphere, generalizes the work of [9] in that it is applicable to a general vorticity field and not just to a field of singular rings or filaments. It also avoids the issue of having to deal with the structure of the image vorticity in order to obtain $\mathbf{u}_I$.

The approach of [2] relies on the observation that the velocity field of the vorticity in the absence of the sphere is potential in the region B occupied by the sphere. Thus, Weiss' formulas for the image of a potential field with singularities outside the sphere is applicable. This directly gives the image potential field $\mathbf{u}_I$ outside the sphere without having to deal with the structure of the image vorticity inside the sphere as in [9]. Thus, as per Knio and Ting, for any point q lying outside the sphere:

$$\begin{aligned} u_I^m(q) \\ = \sum_{i=1}^{N} \frac{-\Gamma_i}{4\pi a} \oint_i \Bigg\langle \mathfrak{Q}_i(\mathbf{l},\mathbf{l}_i) \frac{\partial e_l}{\partial x_m} \\ + \left(-\mathfrak{X}_i(\mathbf{l},\mathbf{l}_i) a^2 x_m + \mathfrak{A}_i(\mathbf{l},\mathbf{l}_i) \left\langle e_{l_i}, \frac{\partial e_l}{\partial x_m} \right\rangle \right) e_l, e_{l_i} \times \mathbf{t}_i \Bigg\rangle ds_i, \\ m = 1,2,3 \end{aligned} \tag{20}$$

where $\mathbf{u}_I = (u_I^1, u_I^2, u_I^3)$, $\mathbf{l} = (x_1, x_2, x_3) \equiv (x,y,z)$ is the position vector of q in a body-fixed frame, $e_l = \frac{\mathbf{l}}{|\mathbf{l}|}$, $e_{l_i} = \frac{\mathbf{l}_i}{|\mathbf{l}_i|}$, a is the radius of the sphere and $\overline{\mathbf{l}}_i = \left|\overline{\mathbf{l}}_i\right| e_{l_i}$ is the position vector of the reciprocal point corresponding to the point on the ith ring with position vector $\mathbf{l}_i$ where $\left|\overline{\mathbf{l}}_i\right| = a^2/|\mathbf{l}_i|$.

The non-dimensional coefficients $\mathfrak{X}_i$,$\mathfrak{Q}_i$ and $\mathfrak{A}_i$ for each ring are defined as follows:

$$\mathfrak{Q}_i(\mathbf{l},\mathbf{l}_i) = \frac{|\bar{\mathbf{l}}|^2}{|\mathbf{l}_i|\,|\bar{\mathbf{l}}-\mathbf{l}_i|\left(\frac{|\bar{\mathbf{l}}-\mathbf{l}_i|}{|\mathbf{l}_i|}+1-\frac{1}{|\mathbf{l}_i|^2}\langle\bar{\mathbf{l}},\mathbf{l}_i\rangle\right)}$$

$$\mathfrak{X}_i(\mathbf{l},\mathbf{l}_i) = \frac{|\mathbf{l}_i|\,|\bar{\mathbf{l}}|}{|\mathbf{l}|^3\,|\bar{\mathbf{l}}-\mathbf{l}_i|^3}$$

$$\mathfrak{A}_i(\mathbf{l},\mathbf{l}_i) = \frac{|\bar{\mathbf{l}}|^3}{|\mathbf{l}_i|^2\,|\bar{\mathbf{l}}-\mathbf{l}_i|}\frac{\frac{|\bar{\mathbf{l}}-\mathbf{l}_i|^2}{|\mathbf{l}_i|^2}+2\frac{|\bar{\mathbf{l}}-\mathbf{l}_i|}{|\mathbf{l}_i|}+1-\frac{\langle\bar{\mathbf{l}},\mathbf{l}_i\rangle}{|\mathbf{l}_i|^2}}{\left(\frac{|\bar{\mathbf{l}}-\mathbf{l}_i|^2}{|\mathbf{l}_i|^2}+\frac{|\bar{\mathbf{l}}-\mathbf{l}_i|}{|\mathbf{l}_i|}\left(1-\frac{\langle\bar{\mathbf{l}},\mathbf{l}_i\rangle}{|\mathbf{l}_i|^2}\right)\right)^2}$$

Note that the reciprocal points always lie inside the sphere. It should also be noted that in the above equations the following identity is used:

$$\frac{\partial}{\partial x_i}\left(\frac{\mathbf{l}}{|\mathbf{l}|}\right) = \frac{\hat{i}}{|\mathbf{l}|} - \frac{x_i}{|\mathbf{l}|^3}\mathbf{l}$$

Regularization of the self-induced field.

Due to the singular nature, i.e. lack of cores, of the N rings there is a well-known singularity in the self-induced velocity field of each ring (see [4] and [7] for more details). A simple though somewhat crude fix to regularize the self-induced velocity field is to use the local induction approximation (see references cited above for the history of this method). Using this approximation the self-induced velocity field of the N rings is obtained as [10]:

$$\mathbf{u}_{SI,i}(q) = \mathbf{u}_{LI,i}(q) := -\frac{\Gamma_i\kappa_i(s_i(q))\log(c_i)}{4\pi}\mathbf{b}_i(s_i(q)),\ q\in C_i, \tag{21}$$

where $\mathbf{t}$, $\mathbf{b}$ and κ represent the unit tangent, unit binormal and curvature fields, respectively, on the rings, and c_i is the cut-off parameter (for the ith ring), assumed constant, appearing in the local induction approximation. Note that the self-induced kinetic energy term obtained by applying the local induction approximation, H_{LI}, satisfies the consistency approximation mentioned earlier i.e $\mathbf{u}_{LI}^n = \frac{\delta H_{LI}}{\delta C_i}\times\mathbf{t}_i$ everywhere on the ith ring (see [10]).

AXISYMMETRIC MODEL

A relatively simple case of the sphere-rings model described above is when the constraint of axisymmetry is imposed. In such an axisymmetric model, the rings are all circles that lie in parallel planes and whose centers pass through the common axis of symmetry which also passes through the center of the sphere, as shown in Figure 2. Each ring in such a model is governed by only two variables, the ring radius and its position along the axis, and the ring evolution equations (5) reduce to ordinary differential equations.

Ring-centered coordinates. To study the axisymmetric sphere-rings model, it is useful to decompose the position vector of the points on the ring into two parts as follows:

$$\mathbf{l}_i(s,t) = -R_i(t)\mathbf{n}_i(s) + z_i(t)\mathbf{b}$$

where $z_i\mathbf{b}$ is the position vector of the center of the i_{th} circular ring with respect to the origin of the body-fixed frame (note that $\mathbf{b}$ is the same for all rings) and $R_i\mathbf{n}_i$ is the vector connecting the center of the i_{th} ring to the points on the i_{th} ring, R_i being the radius of the ring. Note that the principal normals $\mathbf{n}_i$ point towards the center of curvature, i.e. towards the axis of symmetry.

Equations of motion. In the ring-centered coordinates, the term $\mathbf{P}$ in (4) becomes

$$\mathbf{P} = P\mathbf{b}, \tag{22}$$

$$= \left(\sum_{i=1}^{N}\Gamma_i\pi R_i^2 - \frac{a}{2}\sum_{i=1}^{N} I_i(z_i,R_i;a,\Gamma_i)\right)\mathbf{b}, \tag{23}$$

where

$$\int_{\partial B}\mathbf{u}_{V,i}dA =: I_i\mathbf{b} \tag{24}$$

and equation (2) reduces to a relation between scalars

$$L = MU + P, \tag{25}$$

where $L = |\,\mathbf{L}\,|$ and $P = |\,\mathbf{P}\,|$.

Next, we note that due to the axisymmetry of the model, for $j\neq i$,

$$\mathbf{u}_{R,j}\big|_{C_i} = N_{ij}(R_i,R_j,z_i,z_j;\Gamma_i,\Gamma_j,a)\mathbf{n}_i + B_{ij}(R_i,R_j,z_i,z_j;\Gamma_i,\Gamma_j,a)\mathbf{b}$$

Similarly,

$$\mathbf{u}_{I,j}\big|_{C_i} = N_{ij}(R_i,R_j,z_i,z_j;\Gamma_i,\Gamma_j,a)\mathbf{n}_i + B_{ij}(R_i,R_j,z_i,z_j;\Gamma_i,\Gamma_j,a)\mathbf{b}$$

and

$$\nabla\Phi_{B|_{C_i}} = \mathfrak{N}_i(R_i, z_i; \Gamma_i, a)\mathbf{n}_i + \mathfrak{B}_i(R_i, z_i; \Gamma_i, a)\mathbf{b}$$

Note that none of the fields have a tangential component (i.e. swirl).

The equations of motion (13) and (14) for this model reduce to the following system of ODE in the variables (L, R_i, z_i):

$$\frac{dL}{dt} = 0, \tag{26}$$

$$\frac{dz_i}{dt} = \frac{1}{M}\left(-L + \sum_{i=1}^{N}\Gamma_i\pi R_i^2 - \frac{a}{2}\sum_{i=1}^{N} I_i(z_i, R_i; a, \Gamma_i)\right) \tag{27}$$

$$+ \sum_{j,j\neq i}^{N} B_{ij} + \sum_{j=1}^{N} B_{ij} + \mathfrak{B}_i + u_{LI,i}, \tag{28}$$

$$\frac{dR_i}{dt} = -\left(\sum_{j,j\neq i}^{N} N_{ij} + \sum_{j=1}^{N} N_{ij} + \mathfrak{N}_i\right) \tag{29}$$

Dynamic Orbits.

Preliminary simulations of the axisymmetric model, with $N = 2$ and with the ring strengths equal, are shown in Figures 3, 4, 5 and 6. For all the cases shown, $a = 1$, $L = 0$ and the magnitude of the local-induction velocity (21) is set equal to a constant over the instantaneous radius R_i of each ring.

In Figures 3, 5 and 6, the rings start on the same side of the sphere and fairly rapidly settle into a leapfrogging sequence while moving away from the sphere. The sphere velocity at any instant, which is determined by (25), asymptotically reaches a constant value as the rings move away. Moreover, in Figures 5 and 6, both rings pass over the sphere and this passage causes transient oscillations in the sphere velocity. In Figure 3, on the other hand, the rings do not pass over the sphere and the oscillations in the sphere velocity are not seen. In Figure 4, the rings start on opposite sides of the sphere but one ring passes over the sphere and the two rings eventually translate together in a direction opposite to that of the sphere motion.

CONCLUSION AND FUTURE DIRECTIONS

A nonlinear, fully-coupled dynamical model of a neutrally buoyant sphere interacting with thin vortex rings in an inviscid framework is presented in this paper. The Hamiltonian structure and equations of motion follow from those of a more general model [1]. Preliminary simulations of dynamic orbits of the system with the constraint of axisymmetry imposed are also presented.

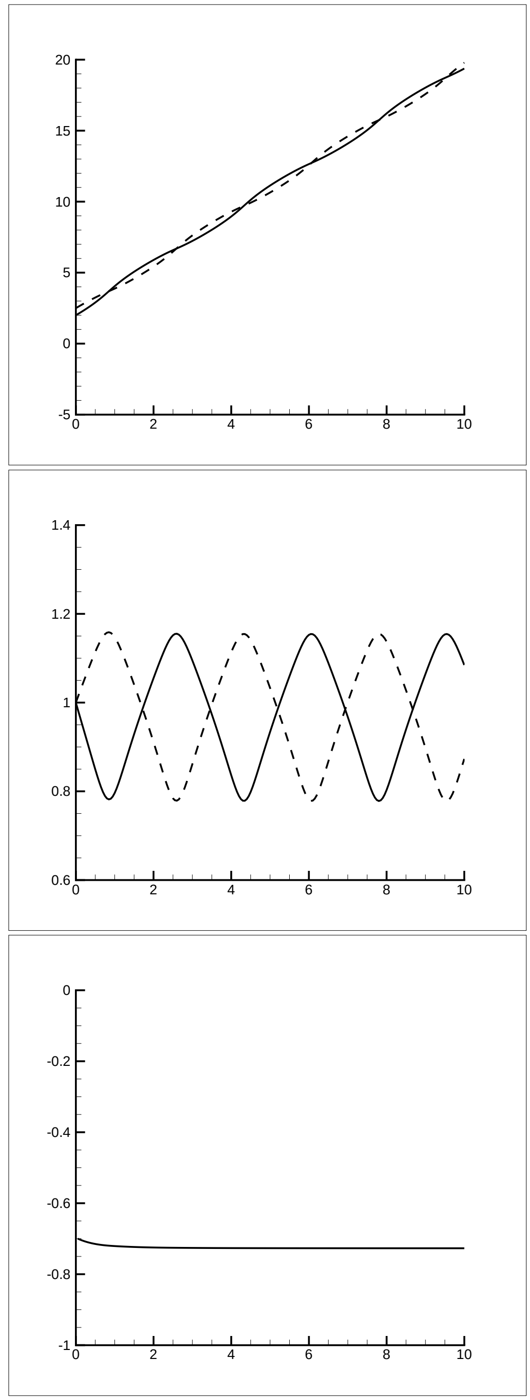

Figure 3. DYNAMIC ORBITS OF TWO AXISYMMETRIC CIRCULAR RINGS AND A NEUTRALLY BUOYANT RIGID SPHERE FOR THE CASE $R_1(0) = R_2(0) = 1$, $z_1(0) = 2, z_2(0) = 2.5$, $\Gamma_1 = \Gamma_2 = 1$ AND $L = 0$. THE TOP FIGURE SHOWS z_i vs. t, THE MIDDLE FIGURE R_i vs. t AND THE BOTTOM FIGURE U vs. t.

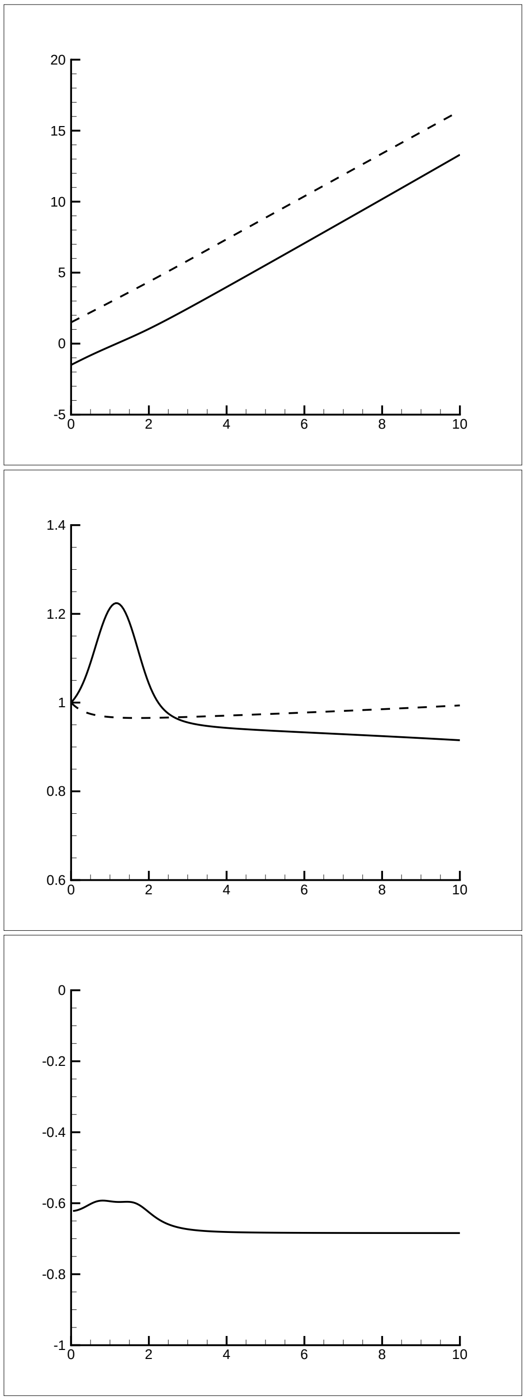

Figure 4. DYNAMIC ORBITS OF TWO AXISYMMETRIC CIRCULAR RINGS AND A NEUTRALLY BUOYANT RIGID SPHERE FOR THE CASE $R_1(0) = R_2(0) = 1, z_1(0) = -1.5, z_2(0) = 1.5, \Gamma_1 = \Gamma_2 = 1$ AND $L = 0$. THE TOP FIGURE SHOWS z_i vs. t, THE MIDDLE FIGURE R_i vs. t AND THE BOTTOM FIGURE U vs. t.

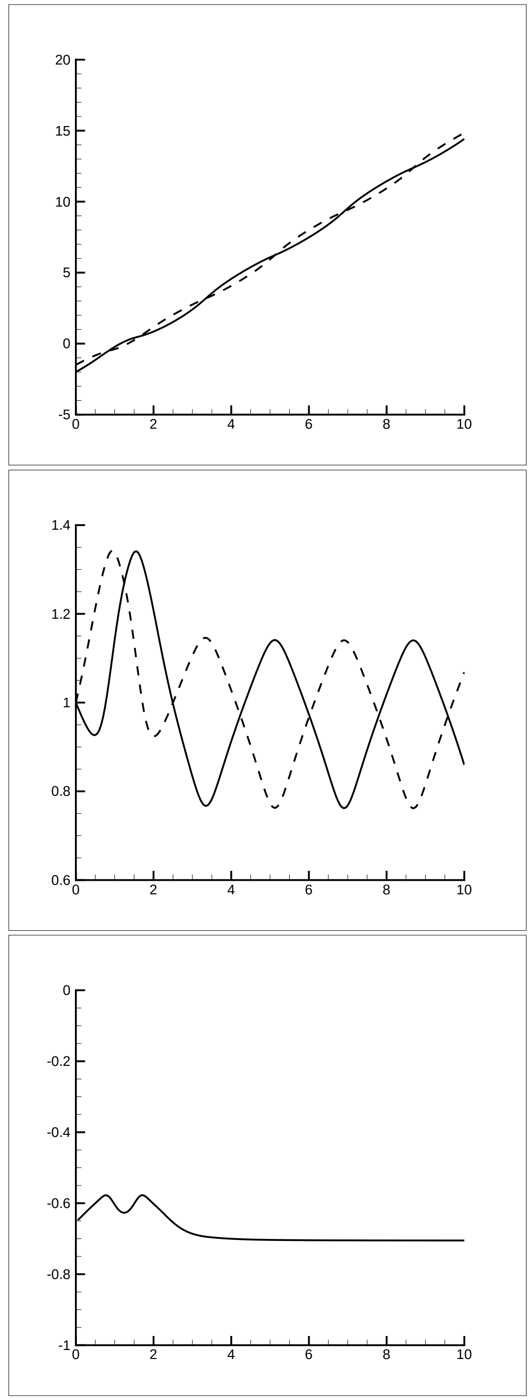

Figure 5. DYNAMIC ORBITS OF TWO AXISYMMETRIC CIRCULAR RINGS AND A NEUTRALLY BUOYANT RIGID SPHERE FOR THE CASE $R_1(0) = R_2(0) = 1$, $z_1(0) = -2.0, z_2(0) = -1.5$, $\Gamma_1 = \Gamma_2 = 1$ AND $L = 0$. THE TOP FIGURE SHOWS z_i vs. t, THE MIDDLE FIGURE R_i vs. t AND THE BOTTOM FIGURE U vs. t.

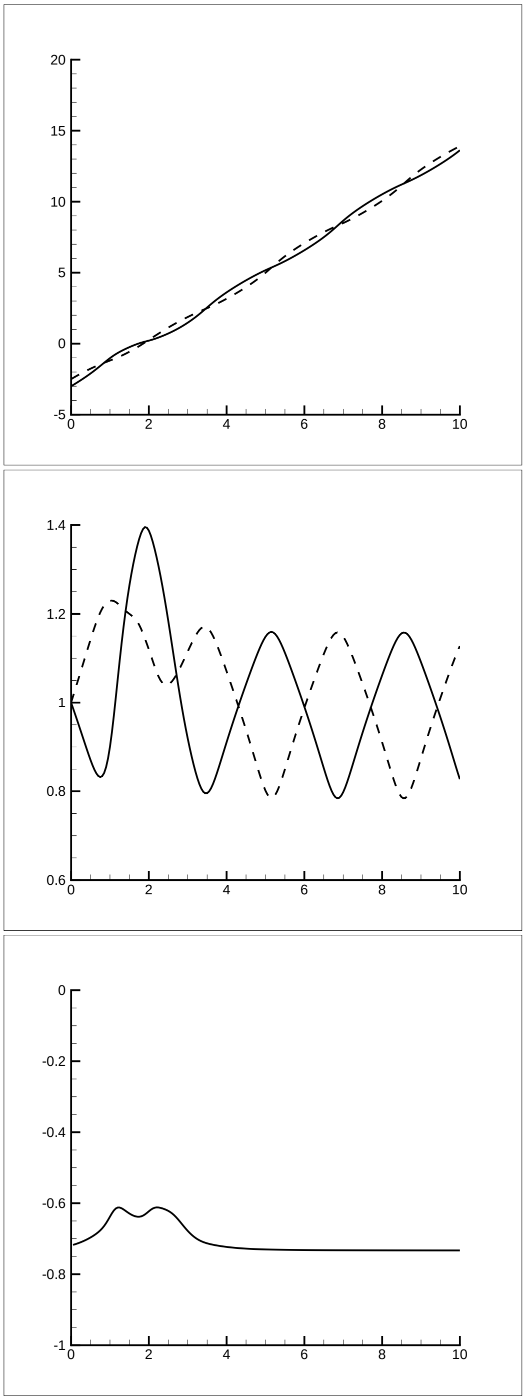

Figure 6. DYNAMIC ORBITS OF TWO AXISYMMETRIC CIRCULAR RINGS AND A NEUTRALLY BUOYANT RIGID SPHERE FOR THE CASE $R_1(0) = R_2(0) = 1$, $z_1(0) = -3.0, z_2(0) = -2.5$, $\Gamma_1 = \Gamma_2 = 1$ AND $L = 0$. THE TOP FIGURE SHOWS z_i vs. t, THE MIDDLE FIGURE R_i vs. t AND THE BOTTOM FIGURE U vs. t.

The qualitative behavior of the dynamic orbits plotted in Figures 3, 4, 5 and 6 is similar to a 2D version of the axisymmetric model considered in [11]. However, as stated earlier, in all these models due to the absence of viscosity certain important dynamical effects can be missed. For example, in the experimental study of [12] in which a vortex ring from a piston cylinder device was made to pass over a neutrally buoyant sphere in a water tank, PIV measurements showed secondary vorticity being shed from the sphere surface as the primary ring passes over it. Thus, a goal for the immediate future is to extend the models presented in this paper to incorporate such viscous effects. Another important goal is to do a more detailed analysis of equilibria, stability and bifurcations as was done in the 2D case (with a circular cylinder and point vortices) in [11] and [13]. For applications to transport phenomena it is also important to extend the models to include many (moving) solid bodies in the flow and, in principle, this extension can be worked out using similar ideas as in this paper. Development of control theoretic models, as in [14] and [15], which can be important for some of the applications mentioned in the Introduction, is another avenue of future research.

REFERENCES

[1] B. N. Shashikanth, A. Sheshmani, S. D. Kelly and J. E. Marsden [2007], Hamiltonian Structure for a Neutrally Buoyant Rigid Body Interacting with *N* Vortex Rings of Arbitrary Shape: The Case of Arbitrary Smooth Body Shape, *submitted to Theoretical and Computational Fluid Dynamics.*

[2] O. M. Knio and L. Ting [1997], Vortical flow outside a body and sound generation, *SIAM J. Appl. Math.*, **57** (4), 972–981.

[3] J. E. Marsden and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics, vol. 17; 1994, Second Edition, 1999*. Springer-Verlag.

[4] P. G. Saffman [1992], *Vortex Dynamics*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge Univ. Press.

[5] L. M. Milne-Thomson [1996], *Theoretical Hydrodynamics*, fifth edition, Dover, New York.

[6] J. E. Marsden and A. Weinstein [1983], Coadjoint orbits, vortices and Clebsch variables for incompressible fluids, *Physica D* **7**, 305–323.

[7] L. Ting and R. Klein [1991], *Viscous Vortical Flows*, Lecture Notes in Physics, **374**, Springer, Berlin.

[8] P. Weiss [1944], On hydrodynamical images: Arbitrary irrotational flow disturbed by a body, *Proc. Cambridge Phil. Soc.*, **40**, 259–261.

[9] M. J. Lighthill [1956], The image system of a vortex el-

ement in a rigid body, *Proc. Cambr. Phil. Soc.*, **52**, 317–321.

[10] B. N. Shashikanth and J. E. Marsden [2003], Leapfrogging vortexrings: Hamiltonian structure, geometricphases and discrete reduction, *Fluid Dynamics Research* **33**, 333–356.

[11] B. N. Shashikanth [2006], Symmetric pairs of point vortices interacting with a neutrally buoyant 2D circular cylinder, *Phys. of Fluids*, **18**, 127103, 17 pages.

[12] J. J. Allen, Y. Jouanne and B. N. Shashikanth [2007], Vortex interaction with a moving sphere, to appear in *J. Fluid Mech.*

[13] B. N. Shashikanth, J. E. Marsden, J. W. Burdick, and S. D. Kelly [2002], The Hamiltonian structure of a 2D rigid circular cylinder interacting dynamically with *N* point vortices, *Phys. of Fluids* **14**, 1214–1227.

[14] Z. Ma and B. N. Shashikanth [2006], Dynamics and control of the system of a 2D rigid circular cylinder and point vortices, *Proc. American Control Conference, Minneapolis, Minnesota, USA, June 14–16, 2006*.

[15] B. N. Shashikanth [2007], Symmetry reduction and control of the dynamics of a 2-D rigid circular cylinder and a point vortex: vortex capture and scattering, *submitted to Euro. J. Control*