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Obtaining a Stable Galerkin ROM in Presence of Shock-Vortex Interactions

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With the increasing need for model-based numerical simulations with computational efficiency and mathematical rigorousness, reduced order models (ROM) derived from Galerkin projection have attracted much attention for the study of their performance in accuracy and stability. Typical POD-Galerkin projection approach computes basis functions from numerical snapshots simulated by high fidelity full order models (FOM). In the current study, snapshots were taken from direct numerical simulation using high-resolution Weighted Essentially Non-Oscillatory (WENO) scheme to solve nonlinear Euler equations. However, severe instabilities were noticed when a ROM derived by typical POD-Galerkin approach was directly applied to the simulation of a supersonic flow passing over a fixed cylinder, where strong shock-vortex interaction appears in downstream flow. To stabilize the POD-Galerkin ROM, we first compared typical stabilization treatments existed in literatures. Then, we proposed a hybrid approach for the improvement of accuracy, efficiency and robustness, and achieved better performance in the current study.

Nomenclature

$\boldsymbol{a}$	Temporal coefficients
c'	Fluctuating part of speed of sound
$\boldsymbol{c}_i$	Eigenvectors of correlation matrix of snapshots
dt	Time step
$\boldsymbol{D}_M$	Diagonal matrix of eigenvalues of the unstable ROM
$\boldsymbol{H}$	Symmetrizing matrix
K	Number of snapshots
M	Reduced order dimension
N	Full order dimension
p	Fluid pressure
$\boldsymbol{q}$	Vector of flow variables
$\bar{\boldsymbol{q}}$	Steady mean of flow variables
$\boldsymbol{q}'$	Fluctuating part of flow variables
$\boldsymbol{R}$	Residual
u	Horizontal component of velocity
$\boldsymbol{u}(t)$	Control input
t	Time
v	Vertical component of velocity
$\boldsymbol{W}$	Correlation matrix of snapshots
$\boldsymbol{y}$	FOM output
$\boldsymbol{y}_M$	ROM output
μ	Stability margin
$\hat{\mu}$	Coefficient of numerical dissipation
τ	Regularization parameter
$\boldsymbol{\Phi}$	Reduced order basis

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ς	Fluid specific volume
γ	Ratio of specific heats
λ	Eigenvalues of the Galerkin system
λ^R	Real part of eigenvalues
λ^I	Imaginary part of eigenvalues
λ^u	Unstable eigenvalues
σ	Eigenvalues of the correlation matrix of snapshots
Subscript	
i, j, k	Mode number indices

I. Introduction

Though high-fidelity full-order models (FOM) provide accurate numerical simulations with high-resolution details for the understanding of physics, there is certainly a large price of slow computational speed from solving the problem in a very high order dimensions including millions/billions number of spatial mesh points and thousands/millions of temporal steps. The excessive computational cost eventually prevents using FOMs in applications such as real-time computation, and optimization in a large parametric space, where reduced-order models (ROM) with lower but acceptable accuracy become attractive.

The idea of model order reduction is to project the equation onto a space with much lower dimension, which is constructed by carefully chosen base functions, and the model equation in the lower dimension is expected to reproduce the results with slightly decreased (but acceptable) accuracy at a much faster speed. One of the most popular approaches for model order reduction is POD-Galerkin projection, using Galerkin projection to construct low order models on the leading modes computed by Proper Orthogonal Decomposition (POD), which captures most of the flow energy by definition.[1] POD-Galerkin models have been applied to many different studies,[2–9] especially after a simple snapshot method was proposed and made a feasible way to handle data with high-resolution in space.[10]

There are always accuracy and stability concerns for ROMs, and studies have been performed on stability of these models for linear systems (or the linearized equations of originally nonlinear systems).[2,8,9,11–13] Using a symmetric definition of inner product instead of the traditional L2 norm has shown a potential to generate POD-Galerkin models with improved stability[2] and higher accuracy.[12] For nonlinear systems, however, simple symmetrization is not sufficient to guarantee stability.[14,15]

In the present study of a supersonic flow passing a fixed cylinder, the problem is highly nonlinear with a strong interaction between shock waves and the shedding vortex street. The momentum transfer by moving vortices results in unsteady shock deformations, which is expected to be the main source of instability in the corresponding ROMs. In a similar case, Lucia and Beran[13] suggested application of numerical dissipation in conjunction with Linear Quadratic Regulation to stabilize the high frequency response of the Galerkin system independently, in order to prevent redundant alteration of eigenvalues corresponding to low frequency response. Their approach works at certain degrees, but the ad hoc way of applying artificial viscosity prevents it from becoming a more general and rigorous approach for other problems. There are two other approaches which seem promising to solve, in a more general way, the instability of ROMs involving unsteady shock waves and complex interactions. Amsallem and Farhat,[8] proposed a stabilization algorithm that is applicable to any ROM regardless of the underlying projection method. Their approach is to search for bases of dimension M in the range of a larger group of bases of dimension $M+r$, by which a stable ROM of dimension M can be generated. The algorithm aims to minimally change the left Reduced Order Basis (ROB) to preserve stability, while keeping the right ROB unaffected for accuracy. Kalashnikova et al.[9] suggested a different approach for linearized equations. Their algorithm involves an eigenvalue reassignment procedure for stabilization and a state feedback concept to minimize deviation of the ROM output from the original FOM output for accuracy control. Both of these methods are applied in the current work for comparison. Correspondingly, we have developed a hybrid approach at the end for better accuracy and efficiency while still generating a stable ROM.

Following sections are arranged to describe the fundamentals and new progresses in section II, which first introduces basic governing equations and idea of using POD-Galerkin projection for construction of ROMs, then reviews the existing treatments for stabilization, and finally proposes a new hybrid approach. In section III, the algorithms are compared for the case of a supersonic flow passing a fixed cylinder and discussions are made on the results from different methods. Final conclusion and comments are in section IV.

II. Methodology

The flow field snapshots were computed by an in-house direct numerical simulation (DNS) code using Weighted Essentially Non-Oscillatory (WENO) scheme for shock capturing and immersed boundary method for solid boundaries and structures.[19] POD modes were computed by the snapshots and used to define the low-dimensional space. Model order reduction using traditional POD-Galerkin projection then projects the original governing equations on to the low-order subspace defined by the leading POD modes and results in a reduced-order model. Unfortunately, for problems involving unsteady shock waves such as those involving interactions with shedding vortices, ROMs constructed by the traditional approach are usually unstable. In this section, two approaches, one by Amsallem and Farhat[8] and the other by Kalashnikova et al.[9] are compared for pros and cons, and a hybrid method is proposed.

A. Governing Equations

Two-dimensional Euler equations are used in simulations and also as the governing equations for Galerkin projection, and they are expressed in conservative form as

$$\frac{\partial \boldsymbol{q}}{\partial t} = -\boldsymbol{A}_1 \frac{\partial \boldsymbol{q}}{\partial x} - \boldsymbol{A}_2 \frac{\partial \boldsymbol{q}}{\partial y}, \tag{1}$$

where

$$\boldsymbol{q} = \begin{pmatrix} u \\ v \\ p \\ \varsigma \end{pmatrix}, \boldsymbol{A}_1 = \begin{pmatrix} u & 0 & \varsigma & 0 \\ 0 & u & 0 & 0 \\ \gamma p & 0 & u & 0 \\ -\varsigma & 0 & 0 & u \end{pmatrix}, \boldsymbol{A}_2 = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & \varsigma & 0 \\ 0 & \gamma p & v & 0 \\ 0 & -\varsigma & 0 & v \end{pmatrix}$$

Defining variables in terms of the mean value and fluctuations, $\boldsymbol{q}(x,y,t) = \bar{\boldsymbol{q}}(x,y) + \boldsymbol{q}'(x,y,t)$, Euler equations are linearized about a steady mean flow,[2,12]

$$\frac{\partial \boldsymbol{q}'}{\partial t} = -\boldsymbol{B}_1 \frac{\partial \boldsymbol{q}'}{\partial x} - \boldsymbol{B}_2 \frac{\partial \boldsymbol{q}'}{\partial y} - \boldsymbol{B}_3 \boldsymbol{q}', \tag{2}$$

where

$$\boldsymbol{B}_1 = \begin{pmatrix} \bar{u} & 0 & \bar{\varsigma} & 0 \\ 0 & \bar{u} & 0 & 0 \\ \gamma \bar{p} & 0 & \bar{u} & 0 \\ -\bar{\varsigma} & 0 & 0 & \bar{u} \end{pmatrix}, \boldsymbol{B}_2 = \begin{pmatrix} \bar{v} & 0 & 0 & 0 \\ 0 & \bar{v} & \bar{\varsigma} & 0 \\ 0 & \gamma \bar{p} & \bar{v} & 0 \\ 0 & -\bar{\varsigma} & 0 & \bar{v} \end{pmatrix},$$

$$\boldsymbol{B}_3 = \begin{pmatrix} \frac{\partial \bar{u}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & 0 & \frac{\partial \bar{p}}{\partial x} \\ \frac{\partial \bar{v}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & 0 & \frac{\partial \bar{p}}{\partial y} \\ \frac{\partial \bar{p}}{\partial x} & \frac{\partial \bar{p}}{\partial y} & \gamma \nabla . \bar{u} & 0 \\ \frac{\partial \bar{\varsigma}}{\partial x} & \frac{\partial \bar{\varsigma}}{\partial y} & 0 & -\nabla . \bar{u} \end{pmatrix},$$

and γ is the ratio of specific heats which is considered to be equal to 1.4 in this paper.

B. Modal Decomposition

POD, a popular approach for modal decomposition, provides an expansion of the original flow data on base functions/modes, which are orthogonal and optimal in energy capturing.[1,3–5,10,16] In the method of snapshots,[10,16] basis functions are generated by multiplying the snapshots by their corresponding weights known as POD coefficients. Computing these coefficients is therefore, equivalent to solving a reduced order eigenvalue problem:

$$\boldsymbol{W} \boldsymbol{c}_i = \sigma_i \boldsymbol{c}_i, \tag{3}$$

where $\boldsymbol{W}$ is a $K \times K$ correlation matrix consisting of inner products of snapshots at different times with K being the number of snapshots.

To reduce the degree of freedom, the original series of *POD expansion* of the original flow field is truncated, while lower but acceptable accuracy is expected for the truncated series to represent the complete space-time function in both its state and dynamic behavior. A typical expansion with truncation for lower dimension is:

$$\boldsymbol{q}(x,y,t)=\bar{\boldsymbol{q}}(x,y)+\sum_{i=1}^{M}\boldsymbol{a}_i(t)\boldsymbol{\Phi}_i(x,y), \tag{4}$$

where M is the number of modes after truncation, $\boldsymbol{\Phi}_i$ is the base functions (i.e. modes), and $\boldsymbol{a}_i$ is a vector of temporal coefficients, which is expected to be "re-computed" by ROM and be similar to the original values directly projected from DNS data.

C. Galerkin Projection

Assuming that the inner product is defined in order to minimize the error of projection on Hilbert space, and based on the fact that this error is equal to the residual, $\boldsymbol{R}$, in modeling the governing equations, in Galerkin projection one is in practice looking for the solution to the following expression based on orthogonality of the POD modes (i.e., projection of the system dynamics on to the basis functions with minimum error):

$$<\boldsymbol{R},\boldsymbol{\Phi}(x,y)>=0 \tag{5}$$

for

$$\boldsymbol{R}=\frac{\partial\boldsymbol{q}}{\partial t}+\boldsymbol{A}_1\frac{\partial\boldsymbol{q}}{\partial x}+\boldsymbol{A}_2\frac{\partial\boldsymbol{q}}{\partial y}. \tag{6}$$

In order to project the nonlinear Euler equations on to the basis functions which in this case are the POD modes, considering the notation in Eq. (7) below:

$$\dot{\boldsymbol{q}}=\boldsymbol{G}(\boldsymbol{q}_1,\boldsymbol{q}_2). \tag{7}$$

Substituting (4) in (7) and taking inner product with each POD mode results in the following ODEs due to orthogonality:

$$\dot{\boldsymbol{a}}_k=<\boldsymbol{G}(\bar{\boldsymbol{q}},\bar{\boldsymbol{q}}),\boldsymbol{\Phi}_k>_{\boldsymbol{H}}+\sum_{i=1}^{M}\boldsymbol{a}_i<\boldsymbol{G}(\boldsymbol{\phi}_i,\bar{\boldsymbol{q}})+\boldsymbol{G}(\bar{\boldsymbol{q}},\boldsymbol{\Phi}_i),\boldsymbol{\Phi}_k>_{\boldsymbol{H}}+ \tag{8}$$

$$\sum_{i,j=1}^{M}\boldsymbol{a}_i\boldsymbol{a}_j<\boldsymbol{G}(\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_j),\boldsymbol{\Phi}_k>_{\boldsymbol{H}},$$

and similarly for the linearized ROM:

$$\dot{\boldsymbol{a}}_k=-\sum_{i=1}^{M}\boldsymbol{a}_i<\boldsymbol{B}_1\frac{\partial\boldsymbol{\Phi}_i}{\partial x}+\boldsymbol{B}_2\frac{\partial\boldsymbol{\Phi}_i}{\partial y}+\boldsymbol{B}_3\boldsymbol{\Phi}_i,\boldsymbol{\Phi}_k>_{\boldsymbol{H}} \tag{9}$$

All the above inner products are constants in time and can be computed off-line.

D. ROM Stabilization

Due to presence of shock-vortex interactions and the associated unsteady oscillations of shock waves in the downstream flow, the original ROM which is formed based on POD-Galerkin projection is highly unstable. Several ROM stabilization approaches are listed below along with our new hybrid approach explained at the end.

(1) Stabilization by adding artificial viscosity similar to the work by Lucia and Beran:[13]

This approach simply introduces numerical dissipation to the model by adding an artificial viscosity term $\hat{\mu}\nabla^2\boldsymbol{q}'$ to the right hand side of the original Euler equations, and ROM is constructed by applying traditional POD-Galerkin projection to the modified Euler equations. The amount of artificial viscosity $\hat{\mu}$ is ideally a small value for stabilization, without bringing too much dissipation to the physical system. The approach is easy to understand and implement. However, the introduced dissipation, even at a low level, still largely affects the accuracy of ROM. Meanwhile, choice of artificial viscosity depends on the system response and is a trial-and-error procedure.

In our test of the approach, extra numerical dissipation brought in by the artificial viscosity affects high frequency components and leads to a "blurred" flow field computed by the ROM, especially near shock waves. Lucia and Beran[13] have suggested to use Linear Quadratic Regulation to target the most stable coefficients independently. However, it still lacks a rigorous connection between the algorithm and accuracy of the system response.

(2) System dynamics modification via a convex optimization problem by Amsallem and Farhat:[8]

This approach takes a different route by rebuilding a new set of basis functions. To stabilize a typical ROM with dimension M, their algorithm suggests to use a larger ROM of dimension $M+r$ to reconstruct a stable ROM with lower dimension M by solving a convex optimization problem. The left ROB of the stable ROM is being searched for in the range of the bases at the dimension of $M+r$, keeping the right ROB unchanged to maintain accuracy. The method is developed for Linear Time Invariant (LTI) systems as:

$$\boldsymbol{E}\frac{d\boldsymbol{x}}{dt}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \tag{10}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t),$$

where $\boldsymbol{x} \in \boldsymbol{R}^N$ is the vector of state variables, $\boldsymbol{u} \in \boldsymbol{R}^S$ is the control input and $\boldsymbol{y} \in \boldsymbol{R}^Z$ is the system output. $\boldsymbol{E} \in \boldsymbol{R}^{N\times N}$, $\boldsymbol{A} \in \boldsymbol{R}^{N\times N}$, $\boldsymbol{B} \in \boldsymbol{R}^{N\times S}$ and $\boldsymbol{C} \in \boldsymbol{R}^{Z\times N}$ are constants in time. The method is applied to the linearized ROM and linear part of the nonlinear ROM in this study for comparison. $\boldsymbol{E}$ is the identity matrix due to non-descriptor form of the current system equations, and $\boldsymbol{u}$ is zero.

Based on Lyapunov stability condition, the equilibrium point $\boldsymbol{x}_0$ of the system in (10) is asymptotically stable if and only if there exists a Symmetric Positive Definite (SPD) matrix $\boldsymbol{P}$ that satisfies the equation below for any SPD matrix $\boldsymbol{Q}$;

$$\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{E} = -\boldsymbol{Q}. \tag{11}$$

Similarly, conforming to the fact that $\lambda + \mu$ is an eigenvalue of the pencil $(\boldsymbol{A} + \mu\boldsymbol{E}, \boldsymbol{E})$ if and only if λ is an eigenvalue of the pencil $(\boldsymbol{A}, \boldsymbol{E})$, it can be shown that the equilibrium point $\boldsymbol{x}_0$ of the system in (10) is asymptotically stable with stability margin μ if and only if there exists a SPD matrix $\boldsymbol{P}$ that satisfies Eq. (12).

$$\boldsymbol{E}^T\boldsymbol{P}(\boldsymbol{A} + \mu\boldsymbol{E}) + (\boldsymbol{A} + \mu\boldsymbol{E})^T\boldsymbol{P}\boldsymbol{E} = -\boldsymbol{Q} \tag{12}$$

Arranging the LTI system (10) in terms of the reduced order variables we have:

$$\boldsymbol{E}_M\frac{d\boldsymbol{x}_M}{dt}(t) = \boldsymbol{A}_M\boldsymbol{x}_M(t) + \boldsymbol{B}_M\boldsymbol{u}(t), \tag{13}$$

$$\boldsymbol{y}_M(t) = \boldsymbol{C}_M\boldsymbol{x}_M(t),$$

where $\boldsymbol{x}_M \in \boldsymbol{R}^M$ is the solution to the ROM, and $\boldsymbol{y}_M \in \boldsymbol{R}^Z$ is the output reconstructed by the ROM. Similarly, $\boldsymbol{E}_M = \boldsymbol{\Phi}_M^T\boldsymbol{E}\boldsymbol{\Phi}_M \in R^{M\times M}$, $\boldsymbol{A}_M = \boldsymbol{\Phi}_M^T\boldsymbol{A}\boldsymbol{\Phi}_M \in R^{M\times M}$, $\boldsymbol{B}_M = \boldsymbol{\Phi}_M^T\boldsymbol{B} \in R^{M\times S}$ and $\boldsymbol{C}_M = \boldsymbol{C}\boldsymbol{\Phi}_M \in R^{Z\times M}$ are constant matrices.

The original method is proposed to be independent of the choice of projection techniques.[8] Since only Galerkin projection is used here, the left and right ROBs are considered the same through out all derivations. Assuming that $\tilde{\boldsymbol{\Phi}}_M$ is the ROB of dimension M that is being searched for in the range of the ROB of a larger ROM of dimension $M+r$, namely $\boldsymbol{\Phi}_{M+r}$, then one can see that $\tilde{\boldsymbol{\Phi}}_M$ can be expressed in terms of a matrix $\boldsymbol{X} \in \boldsymbol{R}^{(M+r)\times M}$ of rank M as follows;

$$\tilde{\boldsymbol{\Phi}}_M = \boldsymbol{\Phi}_{M+r}\boldsymbol{X} \tag{14}$$

Correspondingly, an optimization problem is defined below, based on the Lyapunov stability condition, in a way that it guarantees: a) asymptotic stability of the reduced pencil, b) preserving the basis in case of stability of the original ROM, and c) computational cost in the order of the reduced dimension:

$$min_{\tilde{\boldsymbol{\Phi}}_M \in Range(\boldsymbol{\Phi}_{M+r})} \left\| \tilde{\boldsymbol{\Phi}}_M - \boldsymbol{\Phi}_M \right\| \tag{15}$$

$$\text{s.t.} \quad \boldsymbol{\Phi}_M^T \boldsymbol{E}^T \boldsymbol{\Phi}_{M+r} \boldsymbol{X} \boldsymbol{P} \boldsymbol{X}^T \boldsymbol{\Phi}_{M+r}^T \boldsymbol{A}(\mu) \boldsymbol{\Phi}_M + \boldsymbol{\Phi}_M^T \boldsymbol{A}(\mu)^T \boldsymbol{\Phi}_{M+r} \boldsymbol{X} \boldsymbol{P} \boldsymbol{X}^T \boldsymbol{\Phi}_{M+r}^T \boldsymbol{E} \boldsymbol{\Phi}_M = -\boldsymbol{Q},$$
$$\boldsymbol{P} > \boldsymbol{0}_{M\times M},$$
$$rank(\boldsymbol{X}) = M.$$

Defining $\boldsymbol{M}_E = \boldsymbol{\Phi}_{M+r}^T \boldsymbol{E} \boldsymbol{\Phi}_M$ and $\boldsymbol{M}_A = \boldsymbol{\Phi}_{M+r}^T \boldsymbol{A}(\mu) \boldsymbol{\Phi}_M$, with the change of variables

$$\tilde{\boldsymbol{P}} = \boldsymbol{X} \boldsymbol{P} \boldsymbol{X}^T, \tag{16}$$

a linear equality constraint is constructed as:

$$\boldsymbol{M}_E^T \tilde{\boldsymbol{P}} \boldsymbol{M}_A + \boldsymbol{M}_A^T \tilde{\boldsymbol{P}} \boldsymbol{M}_E = -\boldsymbol{Q} \tag{17}$$

To remove the rank constraint, a SPD matrix $\hat{\boldsymbol{P}} \in \boldsymbol{R}^{(M+r)\times(M+r)}$ is defined in blocks, and assumed to satisfy the Lyapunov equation;

$$\hat{\boldsymbol{P}} = \begin{pmatrix} \hat{\boldsymbol{P}}_{11} & \hat{\boldsymbol{P}}_{12} \\ \hat{\boldsymbol{P}}_{12}^T & \hat{\boldsymbol{P}}_{22} \end{pmatrix}, \tag{18}$$

where $\hat{\boldsymbol{P}}$ is then decomposed and substituted for $\tilde{\boldsymbol{P}}$ in (17);

$$\hat{\boldsymbol{P}} = \begin{pmatrix} \hat{\boldsymbol{P}}_{11} \\ \hat{\boldsymbol{P}}_{12}^T \end{pmatrix} \hat{\boldsymbol{P}}_{11}^{-1} \begin{pmatrix} \hat{\boldsymbol{P}}_{11} & \hat{\boldsymbol{P}}_{12} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0}_{M\times M} & \boldsymbol{0}_{M\times r} \\ \boldsymbol{0}_{r\times M} & \hat{\boldsymbol{P}}_{22} - \hat{\boldsymbol{P}}_{12}^T \hat{\boldsymbol{P}}_{11}^{-1} \hat{\boldsymbol{P}}_{12} \end{pmatrix}. \tag{19}$$

Since $\hat{\boldsymbol{P}}$ and the second term in (19) both satisfy the stability constraint, it follows that the first term also satisfies the Lyapunov equation, therefore;

$$\tilde{\boldsymbol{P}} = \begin{pmatrix} \hat{\boldsymbol{P}}_{11} \\ \hat{\boldsymbol{P}}_{12}^T \end{pmatrix} \hat{\boldsymbol{P}}_{11}^{-1} \begin{pmatrix} \hat{\boldsymbol{P}}_{11} & \hat{\boldsymbol{P}}_{12} \end{pmatrix}, \tag{20}$$

correspondingly, according to the change of variables in (16):

$$\boldsymbol{X} = \begin{pmatrix} \hat{\boldsymbol{P}}_{11} \\ \hat{\boldsymbol{P}}_{12}^T \end{pmatrix}. \tag{21}$$

After some manipulations, the objective function in the optimization problem of Eq. (15) is also arranged in terms of matrix $\hat{\boldsymbol{P}}$:

$$min_{\hat{\boldsymbol{P}} \in R^{(M+r)\times(M+r)}} \left\| \begin{pmatrix} \hat{\boldsymbol{P}}_{12}^T & \hat{\boldsymbol{P}}_{22} \end{pmatrix} \right\|_F + \tau \left\| \hat{\boldsymbol{P}}_{11} \right\|_F, \tag{22}$$
$$\text{s.t.} \quad \boldsymbol{M}_E^T \hat{\boldsymbol{P}} \boldsymbol{M}_A + \boldsymbol{M}_A^T \hat{\boldsymbol{P}} \boldsymbol{M}_E = -\boldsymbol{Q},$$
$$\hat{\boldsymbol{P}} > \boldsymbol{0}_{(M+r)\times(M+r)}.$$

where τ is a regularization parameter. By solving the optimization problem for $\hat{\boldsymbol{P}}$ and computing matrix $\boldsymbol{X}$ via Eq. (21), the system matrices are then modified as:

$$\tilde{\boldsymbol{E}}_M = \boldsymbol{X}^T \boldsymbol{M}_E, \tilde{\boldsymbol{A}}_M = \boldsymbol{X}^T (\boldsymbol{M}_A - \mu \boldsymbol{M}_E), \tag{23}$$
$$\tilde{\boldsymbol{B}}_M = \boldsymbol{X}^T \boldsymbol{B}_{M+r}, \tilde{\boldsymbol{C}}_M = \boldsymbol{C}_{M+r}.$$

The solution to the above optimization problem is obtained in MATLAB by the convex optimization package. One major advantage of this approach is that it is directly applied to the ROMs, and is not restricted to a specific projection method.

However, it is shown (with more details in section III) that all of the eigenvalues of the Galerkin system for the current case are located in the positive real plane, and therefore, a larger modification is required for retaining stability in both of the generated ROMs.

(3) Eigenvalue reassignment with feedback control for accuracy by Kalashnikova et al.:[9]

Considering that instability of the model systems originates from eigenmodes with positive eigenvalues, Kalashnikova et al.[9] proposed an algorithm to reassign eigenvalues to the stable (i.e. negative) region for reconstruction of a stable ROM. Since they included a state feedback control in the process of eigenvalue reassignment to minimize deviation of the output from the modified ROM and the one from the original FOM, the stabilized ROM potentially keeps accuracy in check. Starting with the same LTI system in (13), there is an exact solution,

$$\boldsymbol{x}_M(t) = exp(t\boldsymbol{A}_M)\boldsymbol{x}_M(0) + \int_0^t exp[(t-\epsilon)\boldsymbol{A}_M]\boldsymbol{B}_M\boldsymbol{u}(\epsilon)d\epsilon. \tag{24}$$

Diagonalizing the system dynamics matrix $\boldsymbol{A}_M$ using eigenvalue decomposition, we have:

$$\boldsymbol{A}_M = \boldsymbol{V}_M\boldsymbol{D}_M\boldsymbol{V}_M^{-1} \tag{25}$$

An optimization problem is thus defined for the unstable eigenvalues of the Galerkin system:

$$min_{\lambda_j^u}\sum_{k=1}^{K}\left\|\boldsymbol{y}^k - \boldsymbol{y}_M^k\right\|_2^2, \tag{26}$$

$$\text{s.t.} \quad Re(\lambda_j^u) < 0, j = 1, ..., L,$$

where $\boldsymbol{y}$ is the output of FOM provided in the snapshots and $\boldsymbol{y}_M$ is the ROM output presented in terms of the analytical expression in Eq. (24). Therefore, substituting (24) and (25) for $\boldsymbol{y}_M(t)$ in (13) leads to

$$\boldsymbol{y}_M(t) = \boldsymbol{C}_M[\boldsymbol{V}_M exp(t\tilde{\boldsymbol{D}_M})\boldsymbol{V}_M^{-1}\boldsymbol{x}_M(0) + \int_0^t \boldsymbol{V}_M exp[(t-\epsilon)\tilde{\boldsymbol{D}_M}]\boldsymbol{V}_M^{-1}\boldsymbol{B}_M\boldsymbol{u}(\epsilon)d\epsilon], \tag{27}$$

where $\tilde{\boldsymbol{D}_M}$ is the diagonal matrix of eigenvalues, in which the stable eigenvalues of the original system are maintained and the unstable ones are optimized throughout the minimization problem. After the optimal solution is reached, the system dynamics matrix is recomputed and modified, preserving eigenvectors of the original system:

$$\tilde{\boldsymbol{A}_M} = \boldsymbol{V}_M\tilde{\boldsymbol{D}_M}\boldsymbol{V}_M^{-1} \tag{28}$$

The constrained nonlinear optimization problem in (26) is solved currently by the *fmincon* solver in MATLAB optimization toolbox. In this study we are interested in arranging the eigenvalues in complex conjugate pairs to maintain conformity with the original ROM and reduce computational effort. Therefore, noting that *fmincon* only computes real solutions, the unstable eigenvalues are defined in the following form:

$$\lambda_j = \lambda_j^R + i\lambda_j^I, j = 1:2:L-1 \tag{29}$$

$$\lambda_{j+1} = \lambda_j^R - i\lambda_j^I.$$

Thus, the optimization problem is solved for a total of L unknowns including real and imaginary parts of the eigenvalues, where L is less than or equal to size of the ROM.

Technically, the only constraints required to solve (26) are upper bound constraints on real parts of eigenvalues. However, as the *fmincon* solver computes local minima, due to the large number of unstable eigenvalues, the desired accuracy may not be achievable unless size of the search space is reduced/regularized by adding more constraints. Dealing with this issue becomes more challenging and computationally expensive as the number of modes, along with unstable eigenvalues, increases.

(4) The new hybrid approach: a combination of methods (2) and (3)

It is noticed that algorithm (3) provides a check for accuracy but deteriorates in efficiency and eventually becomes infeasible for problems with increasing number of unstable modes, and algorithm (2), on the other hand, has its problem of accuracy for low frequency components. In the hybrid approach, specific eigenvalues stabilized by algorithm (2) are picked as inputs for the optimization problem in (26), and reduce largely the total number of unstable eigenvalues that are the main obstacle preventing application of approach (3). With the suggested modified inputs provided by algorithm (2), the rest of steps to solve the optimization problem are the same as those described in approach (3), which maintains the accuracy.

The hybrid approach is expected to solve more complex problems, and provide a more efficient and robust stabilization method for ROMs.

III. Results and Discussions

Snapshots are generated by an in-house WENO code for a Mach 2.8 flow passing a fixed cylinder in a two-dimensional channel with solid walls at the top and bottom. A total of 101 snapshots are collected over a grid of N=241,001 points with $dt = 0.01$ and $dx = dy = 0.02$. As shown in figure 1, a standing bow shock is formed in front of the cylinder, and the reflected shock waves from channel walls have strong interactions with the wake vortex structures. Such strong shock-vortex interaction results in highly unsteady shock oscillations that easily destabilize the generated ROMs.

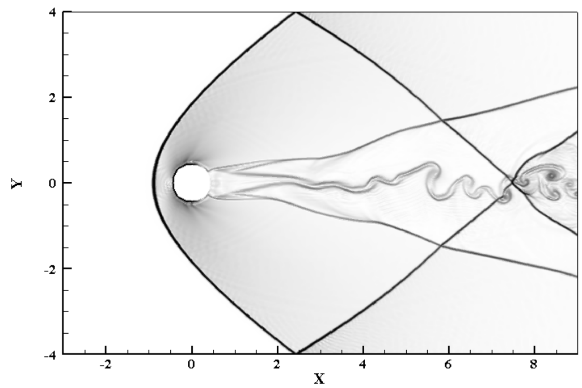


Figure 1: Schlieren image of the computational domain at t=0.58 from the beginning of snapshots collection.

Although FOM is constructed based on the two dimensional nonlinear Euler equations, behavior of both linearized and nonlinear ROMs is investigated here. Both of the ROMs are highly unstable in the sense that all of their eigenvalues are located in the positive real plane.

The approach of Kalashnikova et al.,[9] referred to as Eigenvalue Reassignment (ER) method, is first applied to the 8-mode linearized and nonlinear ROMs. Although like the method of Amsallem and Farhat,[8] this approach is developed for the linear systems, it is applicable to the linear term of nonlinear ROMs as well. The optimization problem is implemented by *fmincon* solver with the interior-point algorithm and default tolerance values. Horizontal component of velocity at a downstream point is depicted as the system output in the objective function. Galerkin coefficients for the 8-mode ROMs stabilized by eigenvalue reassignment are compared against the POD coefficients in figure 2 for both low and high frequency response. Discrepancy of the nonlinear model is reasonable as the optimization is only applied to the linear portion of the Galerkin system in this case. However, the linearized ROM is showing a quite satisfactory resemblance despite the flow field is simulated by a nonlinear FOM (i.e. full Euler equations). It is worth noting that the linearized ROM is originally more unstable than the nonlinear ROM. The ability of the method of Kalashnikova et al.[9] in providing a close agreement by the linearized system is interesting due to the computational savings in modeling highly nonlinear phenomena similar to our physical case, by a linear ROM.

Besides satisfactory results of the ER method in the 8-mode linear model, applying the same method to a 16-mode ROM is computationally inefficient as more constraints are required to obtain the same level of accuracy with 16 unstable eigenvalues.

The method of Amsallem and Farhat,[8] referred to as Convex Optimization (CVX), is applied to the 16 mode ROM with values of $\mu = 0$, $\tau = 10^{-5}$ and $r = 2$. Eigenvalues of the original system and the corresponding stabilized ROM are presented in table 1. The six most unstable modes with eigenvalues λ_1, λ_3, λ_5, and their complex conjugates are stabilized by flipping the sign from positive to negative. Though they are all stabilized, but the large amplitude leads to over-dissipation of those modes and poor accuracy in the low-frequency response. So, for the case of using CVX in the 16-mode ROM, the model is stabilized at the expense of lower accuracy and blurry flow structures.

The purpose of the new hybrid approach proposed in this study is essentially to take the best from both worlds, the efficiency of CVX method and accuracy of Eigenvalue Reassignment method. The idea is to use ER approach only to "fix" those over-dissipated modes (e.g. 3 pairs in the case above) on top of the stabilized ROM provided by the otherwise less-accurate CVX method. Alteration of those modes, first by

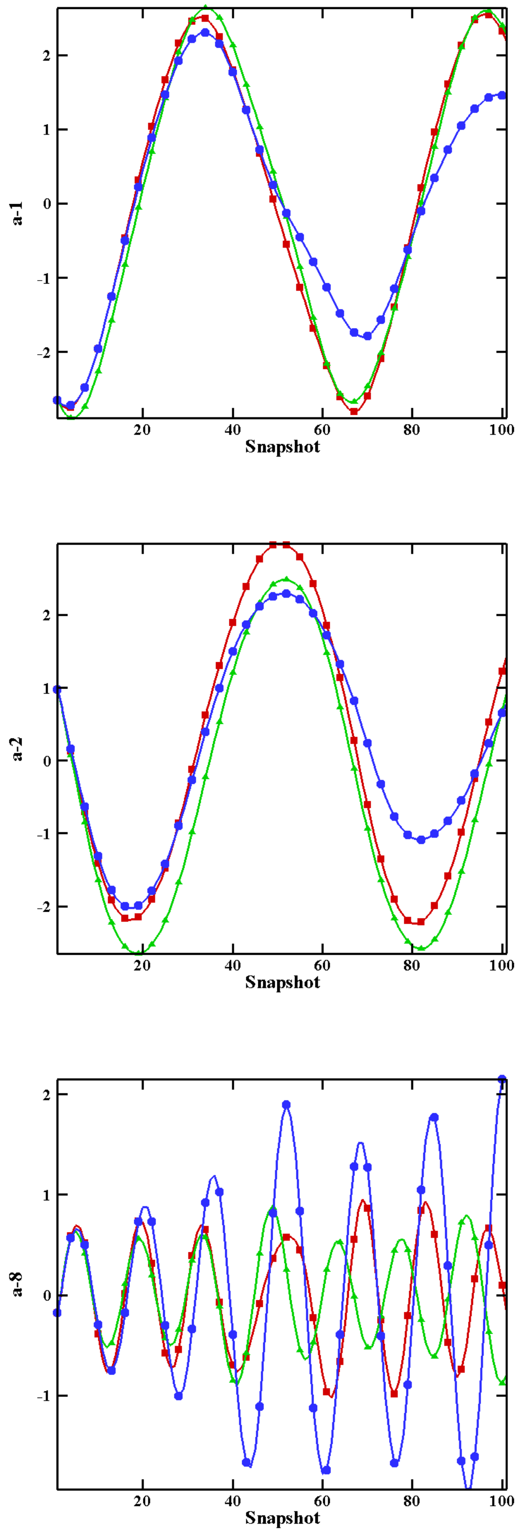


Figure 2: Temporal coefficients of the 8-mode linearized (▲) and nonlinear (●) ROMs stabilized by the original ER method compared against POD coefficients (■).

Table 1: Real parts of the Galerkin system eigenvalues in the linearized ROM with 16 modes.

	Unstable ROM	ROM-CVX
λ_1	2.37651	-2.38557
λ_3	0.88605	-0.89527
λ_5	0.82307	-0.83399
λ_7	0.68800	-0.69961
λ_9	0.53287	-0.54241
λ_{11}	0.38286	-0.09095
λ_{13}	0.28530	-0.29273
λ_{15}	0.17981	-0.18601

CVX method, and then by ER, are clearly shown in figure 3.

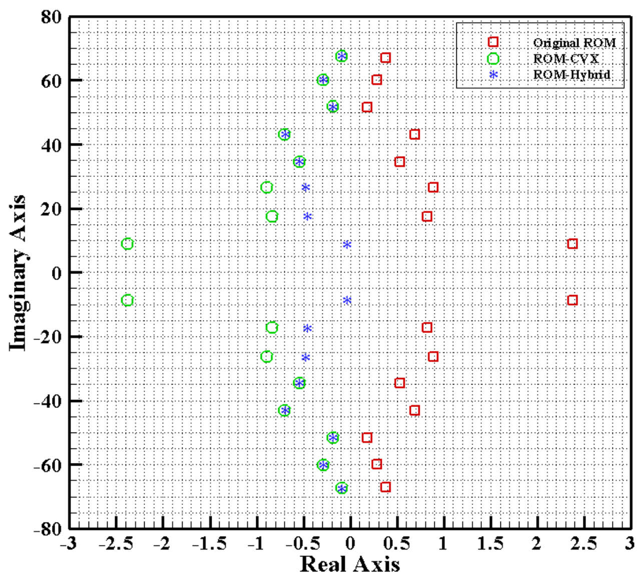
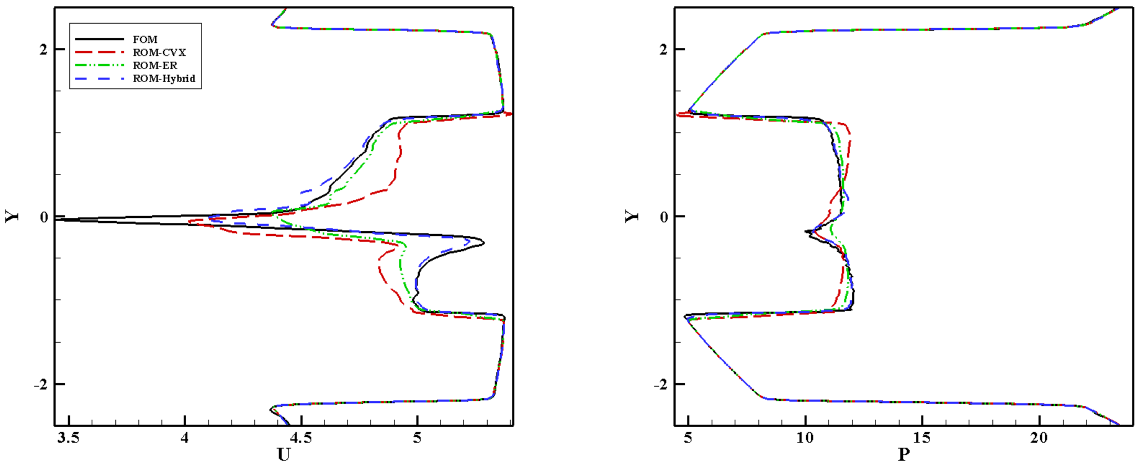


Figure 3: Eigenvalues of the linearized Galerkin system with 16 modes.

Table 2 shows a comparison of performance in terms of the operation of *fmincon* solver in its effort to solve the optimization problem proposed by the original ER method and the CVX-ER hybrid approach. Reducing the number of optimization parameters from 16 to 3 certainly makes the algorithm more efficient and robust. It is noticed that the optimization constraints for (26) are adjusted based on complex conjugate configuration of eigenvalues:

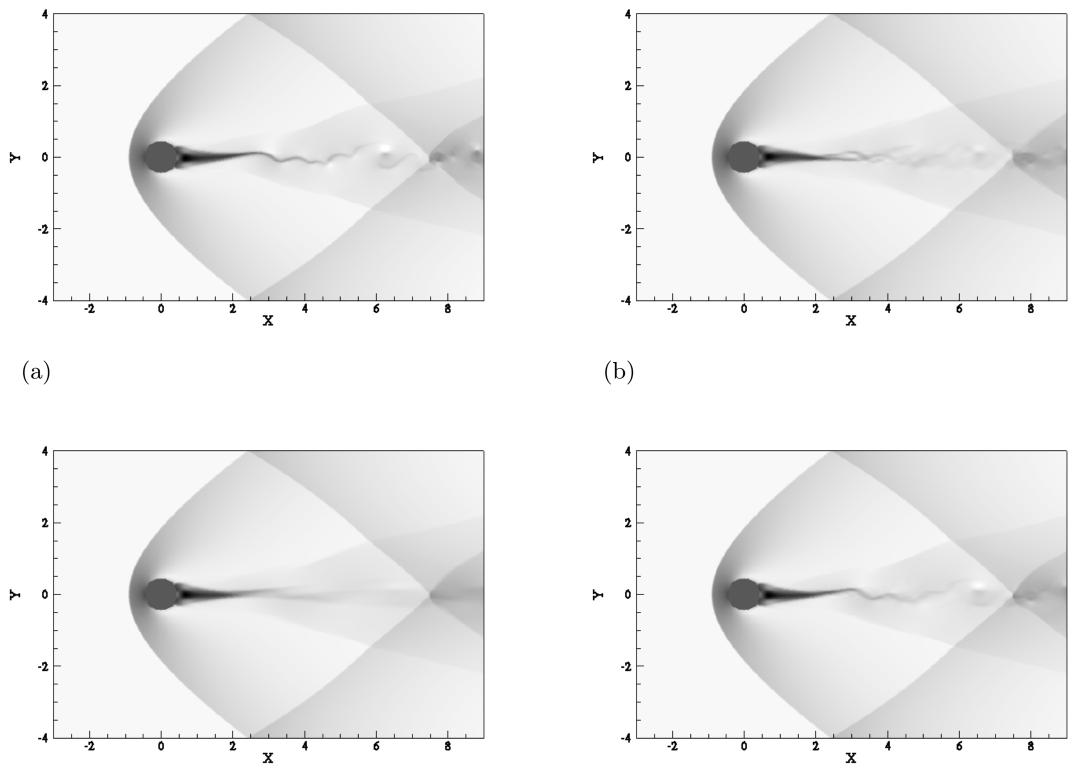
- ER: $Re(\lambda_j^u) < 0, \qquad j = 1, ..., 16$
- Hybrid: $-1 < Re(\lambda_j^u) < 0, \qquad j = 1, ..., 6$

Downstream pressure and velocity profiles at $x = 5$ and $t = 0.66$ are plotted in figure 4 to compare different stabilization methods for a 16-mode ROM with the original DNS results from FOM as benchmark. It is encouraging to find that the results of the ROM stabilized by the hybrid approach are also more accurate in comparison to the original individual approaches. The reconstructed flows shown in figure 5 provide a visual demonstration of the same superior performance in the hybrid approach.



(a) (b)

Figure 4: Profiles of a) horizontal component of velocity and b) pressure, reconstructed by linearized ROMs with 16 modes compared against the FOM.



(a) (b)

(c) (d)

Figure 5: Horizontal component of velocity computed by a) FOM, and 16 mode linearized ROMs stabilized by b) the CVX method, c) the ER method, and d) the new hybrid approach.

Table 2: fmincon performance in stabilization of the 16 mode linearized ROM. Bound constraints in ER are only defined for the real parts of eigenvalues.

	ER	Hybrid
Number of Variables	16	3
Upper-bound constraints	8	3
Lower-bound constraints	0	3
Iterations	22	7
Function evaluations	221	58
First-order optimality	2.142e-07	0.000e+00

Although computational savings provided by the linearized model are quite appealing for the case in hand, development of the hybrid approach for better resemblance with nonlinear ROMs could further extend applications of this method and is a topic of future research.

IV. Conclusion

Nonlinear and linearized ROMs are constructed for supersonic flow past a fixed cylinder in a channel with solid walls, with snapshots provided by numerical simulation of full two-dimensional Euler equations using WENO scheme for shock-capturing. Due to the presence of shock-vortex interactions that lead to unsteady shock oscillations, typical POD-Galerkin ROMs for this problem exhibit large instabilities. Three different treatments are applied for ROM stabilization. Numerical dissipation has been insufficient as a stand-alone method due to over-damping of the high frequency response. The CVX method proposed by Amsallem and Farhat[8] and the ER approach by Kalashnikova et al.[9] are compared for efficiency and accuracy. Inclusion of an explicit check for accuracy in the definition of the objective function in Eigenvalue Reassignment makes it a better choice in terms of the overall accuracy. However, as the number of base modes increases in ROMs, due to the large number of unstable eigenvalues, obtaining an optimal solution is contingent upon increasing the number of constraints. The growth in the number of constraints largely increases computational cost to make the original ER approach inefficient or even computationally infeasible. A CVX-ER hybrid method is developed to largely improve efficiency of the original ER algorithm while maintaining (or even enhancing in some case) the accuracy. Application of the above methods on ROM stabilization for a supersonic flow passing a fixed cylinder in a wall-bounded channel shows superior performance of the hybrid approach on a 16-mode POD-Galerkin ROM.

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