Nodal bubble-tower solutions to radial elliptic problems near criticality

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Abstract

We describe as $\varepsilon \to 0$ radially symmetric sign-changing solutions to the problem

$$-\Delta u = |u|^{\frac{4}{N-2}-\varepsilon} u \quad \text{in } B$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 3$, under zero Dirichlet boundary conditions. We construct radial solutions with k nodal regions which resemble a superposition of "bubbles" of different signs and blow-up orders, concentrating around the origin. A dual phenomenon is described for the slightly supercritical problem

$$-\Delta u = |u|^{\frac{4}{N-2}+\varepsilon} u \quad \text{in } \mathbb{R}^N \setminus B$$

under Dirichlet and fast vanishing-at-infinity conditions.

1 Introduction

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 3$. This paper deals with sign-changing solutions of the Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u = |u|^{q-1} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $q > 1$. It is well known from standard Ljusternik-Schnirelmann theory for even functionals that this problem has infinitely many solutions provided that q is *subcritical*, namely $q < \frac{N+2}{N-2}$. This constraint for existence is optimal since Pohozaev's identity [P] prevents existence of non-trivial solutions if Ω is strictly star-shaped and $q \geq \frac{N+2}{N-2}$. A natural question is that of describing the behavior of these solutions as the exponent grows to critical, namely $q = \frac{N+2}{N-2} - \varepsilon$ with $0 < \varepsilon \to 0$. This behavior has become well understood for the (unique) positive solution u_ε in case that

$$\Omega = B = \{x \ / \ |x| < 1\},$$

the unit ball in $\mathbb{R}^N$, after the work by Brezis and Peletier [BP]. This solution is radial, and its global asymptotic behavior as $\varepsilon \to 0$ turns out to be of the form

$$u_\varepsilon(y) = \gamma_N \left(\frac{1}{1 + [\alpha_0 \varepsilon^{-\frac{1}{2}}]^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_0 \varepsilon^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}}) \tag{1.2}$$

where $\gamma_N = (N(N-2))^{\frac{N-2}{4}}$ and α_0 is a positive constant. In terminology lent from differential geometry, it is commonly said that u_ε develops a *single bubble* at the origin (with blow-up order $\sim \varepsilon^{-\frac{1}{2}}$). This behavior is rather natural since the functions

$$\gamma_N \left(\frac{1}{1 + \mu^{\frac{4}{N-2}} |y - \xi|^2} \right)^{\frac{N-2}{2}} \mu, \quad \xi \in \mathbb{R}^N, \ \mu > 0,$$

correspond precisely to all positive solutions in $\mathbb{R}^N$ of the limiting problem $\Delta w + w^{\frac{N+2}{N-2}} = 0$, see [CGS]. A similar behavior turns out to be true for least-energy solutions in arbitrary domains, as established by Han [H] and Rey [R]. Besides, the asymptotic concentration point can be characterized as the *harmonic center* of the domain, see [W]. More generally positive solutions exhibiting one single bubble show this behavior around critical points of Robin's function (the diagonal of regular part of Green's function), while single bubbling around several points can also be characterized in terms of Green's function, see Bahri, Li and Rey, [BLR]. This also holds for slightly super critical problems, [DFM]. From above the critical exponent, multiple bubbling positive solutions do appear. For $N \geq 4$, and $q = \frac{N+2}{N-2} + \varepsilon$, positive radial solution for the problem

$$\begin{cases} -\Delta u = |u|^{q-1} u + \lambda u & \text{in } B \\ \quad u = 0 & \text{on } \partial B. \end{cases} \tag{1.3}$$

where built at appropriately small $0 < \varepsilon$-dependent range of $\lambda > 0$. These solutions resemble a *tower* of bubbles with different blow-orders. In a related radial problem at the critical exponent involving a parameter dependent weight, solutions of this type were detected in [CL].

The method of construction in [DDM1] can be adapted to certain situations without radial symmetry. For instance, similar phenomena is detected when only symmetry with respect to N axes at a point of the domain is assumed, see [DDM2, DMP]. "Towers of bubbles" in Problem (1.3) actually exist for generic domains, as established in [GJP], see also [FT].

The guiding principle of the present paper is to show that, while typically multiple bubbling of positive solutions at a single point is not expected from the subcritical side, one actually has its presence for *sign-changing solutions.* We shall restrict ourselves to the case of the ball and seek for describing the asymptotic behavior as $\varepsilon \to 0^+$ of radially symmetric solutions to the problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}-\varepsilon} u & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.4}$$

Letting, with some abuse of notation $u(y) = u(r)$ with $r = |y|$, this problem gets reduced to the two-point boundary value problem,

$$\begin{cases} u'' + \frac{N-1}{r} u' + |u|^{\frac{4}{N-2}-\varepsilon} u = 0 & \text{in } (0,1) \\ u'(0) = 0, u(1) = 0. \end{cases} \tag{1.5}$$

Solutions with exactly k zeros in $(0,1)$, namely with k nodal spheres, are unique up to the sign of $u(0)$ as a simple scaling together with uniqueness for the initial value problem shows. Our first result describes asymptotically these solutions, extending the analysis of Brezis and Peletier for the positive one.

Theorem 1.1 *Given $k \geq 1$, for all sufficiently small $\varepsilon > 0$ there is a radial solution $u_{\varepsilon k}$ of (1.4)with exactly k nodal spheres in $(0,1)$, which has the form*

$$u_{\varepsilon k}(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + [\alpha_i \varepsilon^{\frac{1}{2}-j}]^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_i \varepsilon^{\frac{1}{2}-j} (-1)^{i+1} + O(\varepsilon^{\frac{1}{2}})$$

uniformly in B as $\varepsilon \to 0$, for certain positive constants $\alpha_i, i = 1 \ldots k$.

Solutions of (1.4) can be interpreted in terms of bifurcation branches. It is well known that for q sub-critical a bifurcation branch of $k-1$-nodal solutions, $k \geq 1$, stems from each radial eigenvalue λ_k and extends to the left to all $\lambda < \lambda_k$. The result of Theorem 1 describes the intersection of this branch with the axis $\lambda = 0$ as $q \uparrow \frac{N+2}{N-2}$. When $q = \frac{N+2}{N-2}$ some of these branches blow-up before reaching $\lambda = 0$ for dimensions $3 \leq N \leq 6$, moreover there is a $(0,\lambda^*)$ range where no radial sign changing solutions do exist ([ABP1]), while in dimensions 7 of higher they start reaching this axis. We refer the reader to [ABP2, AY, BDE, BN, CFP, CSS, CK1, CK2, GG] for various results along these lines. On the other hand, our result shows that the intersection with the λ-axis as q approaches critical from below has a profile which does not "feel" substantially the change of dimension, a fact not quite obvious a priori.

We do not attempt to give the above result its most general form, but for simplicity of exposition we restrict ourselves to Problem (1.4). We can mention for instance that with similar proof the same result holds for the equation $-\Delta u + u = |u|^{\frac{4}{N-2}-\varepsilon}u$.

The phenomenon described in Theorem 1 has a dual version on the supercritical side in a exterior domain. We consider now the problem

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}+\varepsilon}u & \text{in } \mathbb{R}^N \setminus B \\ \quad u = 0 & \text{on } \partial B, \\ u(y) \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{1.6}$$

In this problem Pohozaev's identity prevents the existence of non-trivial finite energy solutions for exponents less than or equal to critical, namely $\varepsilon < 0$. Instead, if $\varepsilon > 0$, radial solutions with finite number of nodal spheres do exist with a collapsing phenomenon dual to that of Theorem 1 as $\varepsilon \to 0^+$. In this case the k-nodal solution disappears in the form of a "flat" sign-changing tower, as our second result describes.

Theorem 1.2 *Given $k \geq 1$, for all small $\varepsilon > 0$ there exists a radial solution $u_{\varepsilon k}$ with exactly k nodal spheres, of the form*

$$u_{\varepsilon k}(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + [\alpha_i \varepsilon^{i-\frac{1}{2}}]^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_i \varepsilon^{i-\frac{1}{2}} (-1)^{i+1} + O(\varepsilon^{\frac{1}{2}})|y|^{2-N}$$

uniformly in $\mathbb{R}^N \setminus B$, as $\varepsilon \to 0$, for certain positive constants $\alpha_i, i = 1 \ldots k$.

The proof of the above results is carried out in the spirit of singular perturbation methods for nonlinear elliptic problems, in a Lyapunov-Schmidt scheme which traces back at least to Floer and Weinstein's work [FW]. In particular the phenomena described shares a lot with "clustering of spheres" as observed in [MNW] in a different problem. As in [DDM1], our proof takes only partial advantage of the ODE character of the problems and we believe this method may be adapted to general domains under suitable assumptions. We devote the rest of the paper to the Proof of Theorem 1. As it will become apparent below, the proof of Theorem 2 requires just minor variations. On the other hand, we should remark that in non-radial situations it looks difficult to apply variational methods to predict existence in the exterior super-critical problem, while in the interior one subcritical this is entirely standard. The duality sub-supercritical between these problems arises as a rather intriguing matter worth of pursuing.

2 Preliminaries

In all what follows we shall denote $p = \frac{N+2}{N-2}$. Let us consider Problem 1.4 and introduce the change of variables

$$v(x) = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1-\varepsilon}} r^{\frac{2}{p-1}} u(r),$$

where

$$r = e^{-\frac{p-1}{2}x}, x > 0,$$

corresponding to the *Emden-Fowler* transformation, first introduced in [F]. In terms of v the problem becomes

$$\begin{cases} v'' - v + e^{-\varepsilon x}|v|^{p-\varepsilon-1}v = 0 & \text{in } (0,\infty) \\ v(0) = 0, \quad v(x) \to 0 \text{ as } x \to \infty. \end{cases} \tag{2.1}$$

Via a similar transformation, problem (1.6) becomes

$$\begin{cases} v'' - v + e^{-\varepsilon x}|v|^{p+\varepsilon-1}v = 0 & \text{in } (0,\infty) \\ v(0) = 0, \quad v(x) \to 0 \text{ as } x \to \infty. \end{cases} \tag{2.2}$$

Let us consider the *energy functionals* associated to these problems, respectively defined as

$$I^1_\varepsilon(w) = \frac{1}{2}\int_o^\infty |w'|^2 dx + \frac{1}{2}\int_0^\infty |w|^2 dx - \frac{1}{p-\varepsilon+1}\int_0^\infty e^{-\varepsilon x}|w|^{p-\varepsilon+1} dx.$$

and

$$I^2_\varepsilon(w) = \frac{1}{2}\int_o^\infty |w'|^2 dx + \frac{1}{2}\int_0^\infty |w|^2 dx - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{-\varepsilon x}|w|^{p+\varepsilon+1} dx.$$

We will build a suitable ansatz to our problem. To that end consider U solution of

$$\begin{cases} U'' - U + U^p = 0 & \text{in } (-\infty, \infty), \\ U'(0) = 0, U(x) > 0, U(x) \to 0, |x| \to \infty, \end{cases} \tag{2.3}$$

which is given by

$$U(x) = \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-x}(1 + e^{-\frac{4}{N-2}x})^{-\frac{N-2}{2}}.$$

Consider now points $0 < \eta_1 < \ldots < \eta_k$, and call

$$U_i(x) := U(x - \eta_i), \quad V_i := U_i + \pi_i.$$

with $\pi_i(x) := -U(\eta_i)e^{-x}$, and

$$V := \sum_{i=0}^{k} V_i(-1)^{i+1}. \tag{2.4}$$

We seek for solutions of (2.1) and (2.2) of the form $v = V + \phi$ with ϕ small, which going back in the change of variables would give us solutions of (1.4) of the form

$$u(x) \sim \gamma_N \sum_{i=0}^{k} \left(\frac{1}{1 + e^{\frac{4\eta_i}{N-2}} r^2}\right)^{\frac{N-2}{2}} e^{\eta_i}(-1)^{i+1}$$

and for (1.6) a solution of the form

$$u(x) \sim \gamma_N \sum_{i=0}^{k} \left(\frac{1}{1 + e^{-\frac{4\eta_i}{N-2}} r^2}\right)^{\frac{N-2}{2}} e^{-\eta_i}(-1)^{i+1}.$$

In what follows we will focus on proving Theorem 1.1, postponing for the last section Theorem 1.2, pointing out the minor modifications needed.

Before continuing, let us relabel the η_i's conveniently as

$$\eta_1 = -\frac{1}{2}\log\varepsilon + \log\Lambda_1, \quad \eta_{i+1} - \eta_i = -\log\varepsilon - \log\Lambda_{i+1}, \ i = 1, ..., k-1 \tag{2.5}$$

Where the Λ_i's are positive parameters to be determined. We also write $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$. Let us fix a small number $\delta > 0$ and assume the validity of the constraints

$$\delta \le \Lambda_i \le \frac{1}{\delta} \text{ for all } i = 1, \ldots, k. \tag{2.6}$$

Our next step is to carry out asymptotic expansions which are at the core of

a reduction of our problem to that of finding critical points of a functional in a finite dimensional setting.

Lemma 2.1 *Let V be defined as in (2.4). If the η_i's satisfy (2.5) ,then there are numbers $\theta_i, i = 0 \dots 4$, which depend exclusively on N such that*

$$I^1_\varepsilon(V) = k\theta_0 + \varepsilon\Psi_k(\Lambda) - \frac{k^2}{2}\theta_3\varepsilon\log\varepsilon + \theta_4\varepsilon + \varepsilon\beta_\varepsilon(\Lambda) \tag{2.7}$$

where

$$\Psi_k(\Lambda) = \frac{\theta_1}{\Lambda_1^2} + k\theta_3\log\Lambda_1 - \sum_{i=2}^{k}((k-i+1)\theta_3\log\Lambda_i - \theta_2\Lambda_i) \tag{2.8}$$

Besides $\beta_\varepsilon \to 0, \varepsilon \to 0$ uniformly in C^1 norm on the set of Λ_i's that satisfy (2.6).

3 Asymptotic expansions

In this section we prove Lemma 2.1. Below we estimate the terms of the energy functional associated to (2.1), evaluated in $V := \sum_{i=0}^k V_i(-1)^{i+1}$, with η_i's considered to be as before. Now we write

$$I^1_\varepsilon(V) = I^1_0(V) + \varepsilon\left(\frac{d}{d\varepsilon}I^1_\varepsilon\right)_0(V) + o(\varepsilon). \tag{3.1}$$

Let us focus now on the second term of the previous expression, which we call D_ϵ. One has then

$$\begin{aligned}\frac{1}{\varepsilon}D_\varepsilon &= -\frac{1}{(p+1)^2}\int_0^\infty |V|^{p+1}dx + \frac{1}{p+1}\int_0^\infty x|V|^{p+1}dx \\ &\quad + \frac{1}{p+1}\int_0^\infty |V|^{p+1}\log|V|dx.\end{aligned}$$

In this way we can write

$$\begin{aligned}I^1_\varepsilon(V) &= I^1_0(V) - \varepsilon\frac{1}{(p+1)^2}\int_0^\infty |V|^{p+1}dx + \varepsilon\frac{1}{p+1}\int_0^\infty x|V|^{p+1}dx \\ &\quad + \varepsilon\frac{1}{p+1}\int_0^\infty |V|^{p+1}\log|V|dx + o(\varepsilon).\end{aligned}$$

Therefore, we have

$$\begin{aligned} I^1_\varepsilon(V) &= I^1_0(V) - k\varepsilon\frac{1}{(p+1)^2}\int_{-\infty}^{\infty}U^{p+1}dx + \varepsilon\frac{1}{p+1}\left(\sum_{i=1}^k \eta_i\right)\int_{-\infty}^{\infty}U^{p+1}dx \\ &\quad + k\varepsilon\frac{1}{p+1}\int_{-\infty}^{\infty}U^{p+1}\log U dx + o(\varepsilon). \end{aligned}$$

In order to obtain the expected expansion, let us analyze the quantity

$$H_{1,\varepsilon} := (p+1)(I^1_0(V) - \sum_{i=1}^k I^1_0(V_i)).$$

Easily we check

$$\begin{aligned} &\frac{1}{p+1}\left(H_{1,\varepsilon} - \int_0^\infty \left[\sum_{i=1}^k V_i^{p+1} - \left(\left|\sum_{i=1}^k V_i(-1)^{i+1}\right|\right)^{p+1}\right]\right)dx \\ &\qquad = \sum_{i<j}\int_0^\infty (V_i'V_j' + V_iV_j)(-1)^{i+j+2}dx \\ &\qquad = \sum_{i<j}\int_0^\infty (-V_i'' + V_i)V_j(-1)^{i+j+2}dx \\ &\qquad = \sum_{i<j}\int_0^\infty U_i^pV_j(-1)^{i+j+2}dx. \end{aligned}$$

Let us set

$$\mu_1 = 0,\ \mu_l = \frac{1}{2}(\eta_{l-1}+\eta_l),\ l = 2,...,k,\ \mu_{k+1} = \infty.$$

We notice that

$$H_{1,\varepsilon} = -(p+1)\sum_{1\le l\le k, j>l}(-1)^{l+j+2}\int_{\mu_l}^{\mu_{l+1}} V_l^pV_jdx + \rho_1 + \rho_2$$

where

$$\rho_1 := \sum_{l=1}^k\int_{\mu_l}^{\mu_{l+1}}\left[V_l^{p+1} - \left(|V_l(-1)^{l+1} + \sum_{i\neq l}V_i(-1)^{i+1}|\right)^{p+1} + (p+1)\sum_{i\neq l}V_l^pV_i(-1)^{i+l+2}\right]dx.$$

and $\rho_2 := H_{1,\varepsilon} - \rho_1 + (p+1)\sum_{1\le l\le k, j>l}(-1)^{l+j+2}\int_{\mu_l}^{\mu_{l+1}} V_l^p V_j dx$. Since $V_i(x) \le Ce^{-|x-\eta_i|}$ one has

$$\begin{aligned}
|\rho_1| &\le C\sum_{i=1}^{k}\int_{\mu_l}^{\mu_{l+1}}\left(|V_l(-1)^{l+1}+\sum_{i\neq l}V_i(-1)^{i+1}|\right)^{p-1}\left(\sum_{i\neq l}V_i(-1)^{i+1}\right)^2 dx \\
&\le C\int_0^{\frac{\varphi}{2}+\kappa} e^{-(p-1)x}e^{-2|x-\varphi|}dx \\
&\le Ce^{-2\varphi}\int_0^{\frac{\varphi}{2}+\kappa} e^{-(p-3)x}dx \\
&= O(e^{-\frac{p+1}{2}\varphi}) = o(\varepsilon) \quad N\ge 5 \\
&= O(\varepsilon^2|\log\varepsilon|) = o(\varepsilon) \quad N=4 \\
&= O(\varepsilon^2) = o(\varepsilon) \quad N=3.
\end{aligned}$$

Here $\varphi = -\log\varepsilon$ and κ depend only on δ. In the same way we conclude that $\rho_2 = o(\varepsilon)$.Thus the only part of $H_{1,\varepsilon}$ that plays a relevant role is

$$-(p+1)\sum_{1\le l\le k, l\le j}\int_{\mu_l}^{\mu_{l+1}} V_l^p V_j(-1)^{i+j+2}dx$$

$$= -(p+1)\sum_{1\le l\le k, l\le j}\int_{\mu_l}^{\mu_{l+1}}(-1)^l U_l^p(-1)^{l+1}U_{l+1}dx + o(\varepsilon).$$

Setting

$$\tau := (p+1)\sum_{1\le l\le k, l\le j}\int_{\mu_l}^{\mu_{l+1}} U_l^p U_{l+1}dx$$

we find that

$$\tau = (p+1)\sum_{i=1}^{k-1}\int_{\mu_l-\eta_l}^{\mu_{l+1}-\eta_l} U^p(x)U(x-(\eta_{l+1}-\eta_l))dx$$

$$= (p+1)\sum_{i=1}^{k-1}\left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-|\eta_{l+1}-\eta_l|}\int_{-\infty}^{\infty} e^x U^p(x)dx + o(\epsilon)$$

which implies

$$H_{1,\varepsilon} = \theta_2\sum_{l=1}^{k-1} e^{-|\eta_{l+1}-\eta_l|} + o(\varepsilon).$$

After noticing,

$$I_0^1(V_1) = I_0^1(U_1 + \pi_1) = I_0^1(U_1) + DI_0^1(U_1)[\pi_1] + \frac{1}{2}D^2 I_0^1(U_1 + s\pi_1)[\pi_1, \pi_1],$$

for some $s \in (0,1)$, a straightforward computation yields

$$I_0^1(V_1) = \theta_0 + U_1^2(0) + o(\varepsilon).$$

In analogous way,

$$I_0^1(V_i) = \theta_0 + o(\varepsilon),\ \ i \geq 2,$$

where $\theta_0 = \frac{1}{2}\int_{-\infty}^{\infty}(|U'|^2 + U^2)dx - \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}dx$. These facts then yield

$$I_\varepsilon^1(V) = k\theta_0 + \theta_1 e^{-2\eta_1} + \theta_2 \sum_{l=1}^{k-1} e^{-|\eta_{l+1}-\eta_l|} + \theta_3\varepsilon\left(\sum_{i=1}^{k}\eta_i\right) + k\varepsilon\theta_4 + o(\varepsilon).$$

Here,

$$\begin{cases} \theta_0 := \frac{1}{2}\int_{-\infty}^{\infty}(|U'|^2 + U^2)dx - \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}dx \\ \theta_1 := \left(\frac{4N}{N-2}\right)^{\frac{N-2}{2}} \\ \theta_2 := \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} \int_{-\infty}^{\infty} e^x U^p dx \\ \theta_3 := \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}dx \\ \theta_4 := \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\log U dx - \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} U^{p+1}dx. \end{cases}$$

After replacing the above estimates in (3.1), the conclusion follows. □

4 The linearized problem

In this section we consider numbers

$$0 < \eta_1, \ldots, < \eta_k$$

and define

$$Z_i(x) := U_i'(x) - U_i'(0)e^{-x}, i = 1, \ldots, k.$$

Before continuing, let us consider for a moment the problem of finding a function ϕ for which there are constants c_i's, in such a way that

$$\begin{cases} -(V+\phi)'' + (V+\phi) - e^{-\varepsilon x}(|V+\phi|)^{p-\varepsilon-1}(V+\phi) = \sum_{i=1}^{k} c_i Z_i \\ \phi(0) = \lim_{x\to\infty} \phi(x) = 0 \\ \int_0^\infty Z_i \phi dx = 0 \text{ for all } i \le k \end{cases} \tag{4.1}$$

Let us consider the operators

$$L_\varepsilon \phi := -\phi'' + \phi - (p-\varepsilon)e^{-\varepsilon x}|V|^{p-\varepsilon-1}\phi.$$

and

$$L\phi := -\phi'' + \phi - p|V|^{p-1}\phi.$$

Problem (4.1) then gets rewritten as

$$\begin{cases} L_\varepsilon \phi = N_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^{k} c_i Z_i \\ \phi(0) = \lim_{x\to\infty} \phi(x) = 0 \\ \int_0^\infty Z_i \phi dx = 0 \text{ for all } i \le k. \end{cases} \tag{4.2}$$

where

$$N_\varepsilon(\phi) = e^{-\varepsilon x}(|V+\phi|^{p-\varepsilon-1}(V+\phi) - |V|^{p-\varepsilon-1}V - (p-\varepsilon)|V|^{p-\varepsilon-1}\phi)$$

and R_ε is the remainder.

We introduce next a convenient functional analytic setting to analyze our problem. Let us consider the norm

$$||\psi||_* := \inf\{C \mid C\sum_{i=1}^{k} e^{-\sigma|x-\eta_l|} \ge |\psi(x)| \quad \forall x \in (0,\infty)\}.$$

Let us consider the intermediate linear problems

$$\begin{cases} L_\varepsilon \phi = h + \sum_{i=1}^{k} c_i Z_i \\ \phi(0) = \lim_{x\to\infty} \phi(x) = 0 \\ \int_0^\infty Z_i \phi dx = 0 \text{ for all } i \le k. \end{cases} \tag{4.3}$$

and

$$\begin{cases} L\phi = h + \sum_{i=1}^{k} c_i Z_i \\ \phi(0) = \lim_{x\to\infty} \phi(x) = 0 \\ \int_0^\infty Z_i \phi dx = 0 \text{ for all } i \le k \end{cases} \tag{4.4}$$

Lemma 4.1 *Let η_i^n be sequences such that*

$$\eta_1^n \to \infty, \min_{1\le i\le k}(\eta_{i+1}^n - \eta_i^n) \to \infty$$

and ϕ_n a sequence of solutions of (4.4) for $h = h_n$ with $||h_n||_* \to 0$. Then $||\phi_n||_\infty \to 0$.

Proof.

We assume the opposite. Without loss of generality, we assume $||\phi_n||_\infty = 1$. Integrating against Z_l^n

$$\sum_{i=1}^{k} c_i^n \quad \int_0^\infty \quad Z_i^n Z_l^n dx + \int_0^\infty h_n Z_l^n dx$$
$$= \quad \int_0^\infty (-Z_l^{n\prime\prime} + \left[1 - p|V|^{p-1}\right] Z_l^n)\phi_n dx.$$

This leads us to an almost diagonal linear system in the c_i's , therefore invertible for large n. Recall that $Z_l^n(x) + U'(-\eta_i^n)e^{-x}$ is a solution of

$$-Z'' + [1 - pU_l^{p-1}]Z = 0.$$

But we also have $U'(-\eta_i^n)e^{-x} \to 0$ which after an application of dominated convergence yields $\lim c_i^n = 0$. Assume $x_n \ge 0$ is such that $|\phi(x_n)| = 1$. Then, we can further assume that there is an index l and a fixed $M \ge 0$ for which $|\eta_l^n - x_n| \le M$. If we define now

$$-\Phi_n(\cdot) := \phi_n(\eta_l^n + \cdot)$$

we see that up to a subsequence, there is a Φ such that $\Phi_n(\cdot) \to \Phi$, uniformly over compacts and that Φ is a non trivial solution of $-\Phi'' + \Phi - pU^p\Phi = 0$. It follows that for some constant C, $\Phi = CU'$. But from the orthogonality conditions we get

$$\int_{-\infty}^{\infty} U'\Phi dx = 0,$$

which is impossible. This concludes the proof. $\square$

Let us see a direct consequence of the above.

Corollary 4.1 *Under the assumptions of Lemma 4.1 we have that $||\phi_n||_* \to 0$.*

Proof.

If σ is chosen a priori sufficiently small in the definition of the $*-$norm, then

$$|-\phi_n'' + \phi_n| \leq o_n \sum_{i=1}^{k} e^{-\sigma|x-\eta_l|} =: \psi_n(x),$$

with $o_n \to 0$. Thus in a straightforward way one obtain $|\phi| \leq C\psi_n$ for a C large enough. This implies the result.$\square$

We have the validity of the following result.

Proposition 4.1 *There are positive numbers* $\varepsilon_0, \delta_0, R_0$ *and a constant* $C > 0$ *such that if the points* $0 < \eta_1 < \eta_2 \cdots < \eta_k$ *satisfy*

$$R_0 < \eta_1, R_0 < \min_{1\leq i\leq k}(\eta_{i+1} - \eta_i), \eta_k < \frac{\delta_0}{\varepsilon} \tag{4.5}$$

then for all $0 < \varepsilon < \varepsilon_0$ *and all* $h \in C[0,\infty)$ *with* $||h||_* < \infty$*, problem (4.4) admits a unique solution that we shall call* $T(h) := \phi$*, which besides satisfies*

$$||T(h)||_* \leq C||h||_*, \text{ and } |c_i| \leq C||h||_*.$$

Proof.

Let us consider the Hilbert space

$$H = \{\phi \in H_0^1(0,\infty); \int_0^\infty Z_i\phi = 0, \forall i = 1,\ldots,k\}$$

endowed with its natural inner product. Then it is possible to rewrite problem (4.4) in this space, thanks to Riesz representation theorem, in the form:

$$\phi = K_0\phi + \hat{h}.$$

where $\hat{h}$ is linear and K_0 is a compact operator. Fredholm's alternative guarantees unique solvability of this problem provided the homogeneous equation has only the trivial solution. This latter statement can be deduced indirectly for $R_0, \varepsilon_0, \delta_0$ chosen properly, assuming the opposite would lead us to a contradiction with the previous lemma. Besides, continuity can be derived in a similar manner. $\square$

The previous analysis lead to the main result of this section,

Proposition 4.2 *There are positive numbers $\varepsilon_0, \delta_0, R_0$ and a constant $C > 0$ such that if the points $0 < \eta_1 < \eta_2 \cdots < \eta_k$ satisfy*

$$R_0 < \eta_1, R_0 < \min_{1 \leq i \leq k} (\eta_{i+1} - \eta_i), \eta_k < \frac{\delta_0}{\varepsilon} \tag{4.6}$$

then for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C[0, \infty)$ with $||h||_* < \infty$, problem (4.3) admits a unique solution that we will denote $T_\varepsilon(h) := \phi$, which besides satisfies

$$||T_\varepsilon(h)||_* \leq C||h||_*, \text{ and } |c_i| \leq C||h||_*.$$

Proof.

Defining $\mathcal{P}(\phi) := T(h + ((p-\varepsilon)|V|^{p-1-\varepsilon} - p|V|^{p-1})\phi)$, we see $\mathcal{P}$ is a contraction for small ε, thus a standard fixed point argument gives us existence. Now, we see (4.3) can be rewritten

$$\begin{cases} L\phi = h + ((p-\varepsilon)|V|^{p-1-\varepsilon} - p|V|^{p-1})\phi + \sum_{i=1}^k c_i Z_i \\ \phi(0) = \lim_{x\to\infty} \phi(x) = 0 \\ \int_0^\infty Z_i \phi dx = 0 \quad \forall i \leq k. \end{cases} \tag{4.7}$$

From the previous proposition, we conclude

$$||\phi||_* \leq ||((p-\varepsilon)|V|^{p-1-\varepsilon} - p|V|^{p-1})\phi||_* + C||h||_*.$$

But the first term on the right hand side is $o(1)||\phi||_*$, hence we get for small ε

$$||\phi||_* \leq 2C||h||_*$$

and the proposition is proven. $\square$

Next we study some differentiability properties of T_ε with respect to the η_i's that will be important for later purposes. Let us consider the Banach Space

$$\mathcal{C}_* = \{f \in \mathcal{C} \mid ||f||_* < \infty\}$$

and $\mathcal{L}(\mathcal{C}_*)$ the space of linear operators of $\mathcal{C}_*$.

Our purpose is to prove

Lemma 4.2 *Under the conditions of the previous proposition, the map $\eta \to T_\varepsilon$, with values in $\mathcal{L}_\varepsilon(\mathcal{C}_*)$ is of class C^1. Moreover, there is a constant $C > 0$ such that*

$$||\nabla_\eta T_\varepsilon||_{\mathcal{L}_\varepsilon(\mathcal{C}_*)} \leq C$$

uniformly with respect to the vectors η that satisfy (4.5).

Proof.

Let us recall that ϕ satisfies the equation

$$L_\varepsilon \phi = h + \sum_{i=1}^{k} c_i Z_i,$$

plus the orthogonality conditions. Fix $l \in \{1, \ldots, k\}$. Define for $j \in \{1, \ldots, k\}$, constants b_j's such that

$$b_j \int_0^\infty |Z_j|^2 dx + \int_0^\infty \nabla_{\eta_l} \phi Z_l dx + \sum_{i \neq j} b_i \int_0^\infty Z_i Z_j dx = 0.$$

Once again, this turns out to be an almost diagonal system, therefore we can define also

$$f := \sum_{i=1}^{k} b_i L_\varepsilon Z_i + c_l \partial_{\eta_l} Z_l + (p - \varepsilon) e^{-\varepsilon x} \partial_{\eta_l} (|V|^{p-1-\varepsilon}) \phi$$

Then $\partial_{\eta_l} \phi$ satisfies

$$\nabla_{\eta_l} \phi = T_\varepsilon(f) - \sum_{i=1}^{k} b_i L_\varepsilon Z_i.$$

Moreover, $||f||_* \leq C||h||_*, |b_l| \leq C||\phi||_*$. Besides one has $\partial_{\eta_l} \phi$ depends continuosly on the $\eta_i' s$ and with respect to h in this norm. Ultimately

$$||\nabla_{\eta_l} \phi||_* \leq C||h||_*$$

and the result follows. $\square$

In what follows we will assume the validity of the constraints

$$\begin{cases} \eta_1 > \frac{1}{2} \log(M\varepsilon)^{-1}, \log(M\varepsilon)^{-1} < \min_{1 \leq i \leq k} (\eta_{i+1} - \eta_i) \\ \eta_k < k \log(M\varepsilon)^{-1}. \end{cases} \tag{4.8}$$

For our later purposes it is enough to consider $||\phi||_* \leq \frac{1}{4}$. One easily verifies that under these conditions, provided that σ is fixed small enough, that

$$||N_\varepsilon(\phi)||_* \leq C||\phi||_*^{\min\{p,2\}} \text{ y } ||R_\varepsilon||_* \leq C\varepsilon^{1-\sigma}. \tag{4.9}$$

We have the validity of the following fact.

Proposition 4.3 *Assume (4.8) holds. Then there is a constant $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon_0 > \varepsilon > 0$, there is a unique solution $\phi = \phi(\eta)$ to problem (4.2) , which besides satisfies*

$$||\phi||_* \leq C\varepsilon^{1-\sigma}.$$

Moreover, the function that $\eta \mapsto \phi(\eta)$ is of class C^1 for $||\cdot||_*$. In fact one has

$$||\nabla_\eta \phi||_* \leq C\varepsilon^{1-\sigma}.$$

Proof.

Define $A_\varepsilon(\phi) := T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon)$ We see then that solving (4.2) is equivalent to the problem of finding ϕ such that $A_\varepsilon(\phi) = \phi$. For this matter, it suffices to show that A_ε is a contraction in a proper region. Let

$$\mathcal{F}_r := \{\phi \in C[0,\infty) \mid ||\phi||_* \leq r\varepsilon^a\}$$

Where r is a positive number that will be fixed later, and $a \in (0,1)$. From proposition 2 and from (4.9) we get

$$||A_\varepsilon(\phi)||_* \leq C||N_\varepsilon(\phi) + R_\varepsilon||_* \leq C(r\varepsilon^{\min\{p,2\}} + \varepsilon^{1-\sigma}) \leq r\varepsilon^a,$$

for ε small enough and large r , chosen independently from ε. Thus $A_\varepsilon(\mathcal{F}_r) \mapsto \mathcal{F}_r$. But as N_ε is a contraction in the norm $||\cdot||_*$, one gets A_ε is one as well. Regarding differentiability, in a canonical way one proves

$$||\nabla_\eta \phi||_* \leq C(||N_\varepsilon(\phi) + R_\varepsilon||_* + ||\nabla_\eta N_\varepsilon(\eta,\phi)||_*) \leq C\varepsilon$$

and the proof is concluded. □

5 The finite dimensional reduction

In this section we fix M large and assume conditions (4.8) holds for the vector η. In the previous sections we have achieved reducing our problem to that of finding η such that the $c_i's$ that appear in (4.2) are all zero. This would make $V + \phi$ satisfy our equation. More precisely we need to solve the linear system

$$c_i(\eta) = 0, \quad \text{for all } i = 1, \ldots, k. \tag{5.1}$$

This problem is equivalent to a variational one. Let us consider

$$\mathcal{J}_\varepsilon = I^1_\varepsilon(V + \phi), \tag{5.2}$$

with $\phi = \phi(\eta)$ given by the last proposition. We claim that solving (2.1) is equivalent to finding critical points of this functional. To see this, let us notice first that by definition of I_ε and ϕ, one has

$$DI_\varepsilon(V+\phi)[Z_i] = 0 \quad \text{ for all } i = 1, \ldots, k. \tag{5.3}$$

Now, we easily check that $\nabla_{\eta_i}(V+\phi) = Z_i + o(1)$ with $o(1) \to 0$ as $\varepsilon \to 0$ with respect to $||\cdot||_*$. But we see that the lower order term can be decomposed as the sum of an element spanned by the Z_i and a β in its orthogonal. From the definition of ϕ we see again $DI_\varepsilon(V+\phi)[\beta] = 0$. In this way (5.1) is the same as solving

$$\nabla \mathcal{J}_\varepsilon(\eta) = 0.$$

To get this, let us show first

Lemma 5.1 *One has the validity of the following*

$$\mathcal{J}_\varepsilon(\eta) = I_\varepsilon(V) + o(\varepsilon),$$

where $o(\varepsilon)$ is uniform in the C^1 sense, over all η satisfying conditions (4.8), for a given M.

Proof.

Noticing that

$$I^1_\varepsilon(V+\phi) - I^1_\varepsilon(V+\phi) := \int_0^1 tD^2 I^1_\varepsilon(V+t\phi)dt.$$

and since $||\phi||_* = O(\varepsilon)$, we obtain

$$\mathcal{J}_\varepsilon(\eta) - I^1_\varepsilon(V) = O(\varepsilon^2).$$

In a similar way we see that

$$\nabla_\eta(\mathcal{J}_\varepsilon(\eta) - I^1_\varepsilon(V)) = o(\varepsilon).\square$$

6 Proof of the main results

As it was natural to expect, the solutions of (2.1) we seek for, correspond to critical points of a certain functional related to Ψ_k. Before going into the actual proof of theorem 1.1, let us notice that Ψ_k attains a global non-degenerate minimum at the point

$$\bar{\Lambda} = (\sqrt{\frac{2\theta_1}{k\theta_3}}, (k-1)\frac{\theta_3}{\theta_2} \dots \frac{\theta_3}{\theta_2})$$

in $\mathbb{R}_+^K$. From what we know so far this is almost all what we need. It only remains to collect what was done in the previous sections to conclude.

Proof of Theorem 1.1. We just need to find a critical point of $\mathcal{J}_\varepsilon(\eta)$ where $\eta = \eta(\Lambda)$, is given by the choice (2.5). Then, thanks to Lemma 5.1,

$$\varepsilon^{-1}\nabla\mathcal{J}_\varepsilon(\eta(\Lambda)) = \nabla\Psi_k(\Lambda) + o(1)$$

where $o(1) \to 0$ uniformly over Λ satisfying $\frac{1}{M} < \Lambda_i < M$. Now, since the critical point $\bar{\Lambda}$ of Ψ_k is non-degenerate, it follows that $deg(\nabla\Psi_k, \mathcal{N}, 0) \neq 0$ for all $\mathcal{N}$ small neighborhood of $\bar{\Lambda}$. Finally, going back in the change of variables, one obtain the solutions of the original problem, of the desired form. $\square$

Proof of Theorem 1.2. It is enough to notice that the term D_ε in (3.1) differs from that we found in the expansion of Lemma 2.1 slightly, being actually meaningless when comes to the main functional. Actually if we change θ_4 to be $\theta_4 := -(\frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\log U dx - \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} U^{p+1} dx)$ instead, the result remains valid, with Ψ_k leading the expansion at first order. Besides, the linear problem can be worked through in the same way. Moreover, operator $\mathcal{L}$ in Lemma 4.1 is the same for both problems. The conclusion of Proposition 4.2 is unaltered if we set $\mathcal{P}$ in the obvious way. Finally, differentiability properties can be derived in a similar way. Critical points to our problem translate in what stated at the beggining, after going back in the change of variables, thus yielding Theorem 1.2. $\square$

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