GAMMA-CONVERGENCE AND THE EMERGENCE OF VORTICES FOR GINZBURG-LANDAU ON THIN SHELLS AND MANIFOLDS

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Abstract. We analyze the Ginzburg-Landau energy in the presence of an applied magnetic field when the superconducting sample occupies a thin neighborhood of a bounded, closed manifold in $\mathbb{R}^3$. We establish Γ-convergence to a reduced Ginzburg-Landau model posed on the manifold in which the magnetic potential is replaced in the limit by the tangential component of the applied magnetic potential. We then study the limiting problem, constructing two-vortex critical points when the manifold $\mathcal{M}$ is a simply connected surface of revolution and the applied field is constant and vertical. Finally, we calculate that the exact asymptotic value of the first critical field H_{c1} is simply $\big(4\pi/(\text{area of }\mathcal{M})\big)\ln\kappa$ for large values of the Ginzburg-Landau parameter κ. Merging this with the Γ-convergence result, we also obtain the same asymptotic value for H_{c1} in 3d valid for large κ and sufficiently thin shells.

1. Introduction

We initiate here an investigation of the behavior of thin superconducting shells when subjected to an applied magnetic field within the context of Ginzburg-Landau theory. One goal is to identify, via the theory of Γ-convergence (cf. [8]), an appropriate limiting energy as the thickness of the shell approaches zero. This leads us to the problem of Ginzburg-Landau on a closed manifold. A second major thrust of this study is to identify the so-called first critical field, H_{c1}, representing the threshold in applied field strength that must be crossed in order to first see vortices in minimizers of the energy. We identify this value in the asymptotic regime

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of large Ginzburg-Landau parameter, both for the limiting problem on a manifold and for Ginzburg-Landau on thin 3d shells.

Within the physics community, there are numerous studies of the response of a spherical superconducting shell or film to a magnetic field, including the experimental study [25] and the theoretical studies [9], [20] and [26]. The latter are primarily computational studies using a Ginzburg-Landau theory based on the presence of a magnetic monopole located at the center. (The monopole assumption, which we do not invoke, leads to the condition that the magnetic field strength is uniform throughout the surface of the sphere.) Within the applied mathematics community, we note the computational work in [10] and [11] on superconducting spheres in the presence of a vertical magnetic field. Here the authors capture various vortex patterns on the surface of the sphere as the magnetic field strength is varied. Note that all of the research cited above focuses solely on a spherical geometry and is largely computational. Two primary aims of our research here are to inject some rigorous mathematical analysis into the discussion and to explore the role that the geometry and topology of the limiting manifold may play in generating non-trivial vortex behavior.

While, to our knowledge, there has been little rigorous analysis of Ginzburg-Landau for thin shells or on a closed manifold, there has of course been extensive work on the thin film limit of Ginzburg-Landau as a 3d sample collapses to a bounded, planar domain with boundary. In [2], the authors show that when the applied field is vertical, then in this thin film limit, minimizers of the 3d problem approach a function of two variables that solves a reduced Ginzburg-Landau system in which the magnetic potential is replaced by the applied magnetic potential corresponding to the applied field. We also obtain a reduced problem, though now it is on a manifold and our main result in this direction, Theorem 3.1, consists of proving the full Γ-convergence of the 3d Ginzburg-Landau energy to a 2d energy, valid for any fixed value of the Ginzburg-Landau parameter κ. Thus, in addition to showing that sequences of minimizers approach a minimizer of the limiting energy (and hence, in particular, approach a solution to the limiting Euler-Lagrange system), this opens the door to the future study of existence and convergence of local minimizers and critical points through the machinery of

Γ-convergence, in the spirit of [15, 16, 18, 19], and to the study of associated gradient flows in the sense of [23].

Loosely stated, for $\Omega_\varepsilon \subset \mathbb{R}^3$ representing an ε-neighborhood of a smooth, compact manifold $\mathcal{M}$, we begin by showing in Section 3 that the sequence of Ginzburg-Landau energies (cf. [24])

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left(|(\nabla - i\mathbf{A})\Psi|^2 + \frac{\kappa^2}{2}\left(|\Psi|^2-1\right)^2\right)dX + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\nabla\times\mathbf{A}-\mathbf{H}^e|^2\,dX$$

Γ-converges in an appropriate topology to the energy

$$\int_{\mathcal{M}}\left(|(\nabla_{\mathcal{M}} - i(\mathbf{A}^e)^\tau)\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\right)d\mathcal{H}^2_{\mathcal{M}}(x).$$

(See conditions (3.5) and (3.9) below for a working definition of Γ-convergence.) Here $\Psi : \Omega_\varepsilon \to \mathbb{C}$ is the order parameter in 3d and $\psi : \mathcal{M} \to \mathbb{C}$ is the corresponding object defined on the manifold. The vector field $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is the effective magnetic potential, $\mathbf{H}^e : \mathbb{R}^3 \to \mathbb{R}^3$ is the given applied magnetic field, assumed to be divergence-free and smooth but otherwise general, and $(\mathbf{A}^e)^\tau$ is the tangential component of the restriction of the applied magnetic potential to the manifold $\mathcal{M}$. (More precise definitions and notational explanations are given in Section 2.) The scaling factor of $\frac{1}{\varepsilon}$ in the 3d energy is chosen so as to keep the energy of minimizers bounded but non-zero in the limit. We should note that in this article, we assume the collapse of the 3d domains Ω_ε to $\mathcal{M}$ is uniform but one can surely adjust our proof of Γ-convergence to handle variable thickness approach to $\mathcal{M}$ as has been considered in various studies of planar thin superconducting films subjected to vertical applied fields, e.g. [5, 6, 17]. The replacement of $\mathbf{A}$ by $(\mathbf{A}^e)^\tau$ in the Γ-limit is a manifestation of the fact that for thin samples, the magnetic field penetrates the sample to leading order in the thickness.

We stress that the Γ-convergence result established here is valid for any smooth, compact manifold, with or without boundary, and any smooth divergence-free applied field. In this level of generality, we also demonstrate the existence of a positive first critical field below which minimizers do not vanish, cf. Proposition 3.7.

Having derived the Γ-limit, we turn to an analysis of the limiting energy, a variational problem which is quite interesting in its own right. Recall that for the 2d planar version of this problem subject to Neumann boundary conditions, a vortex might escape through the boundary, but this phenomenon is precluded if we take our compact manifold to be without boundary. This restriction should lead to interesting new aspects of vortex dynamics and though we do not pursue the time-dependent problem here, we do exploit a related feature of this problem, namely the fact that if the order parameter ψ vanishes on $\mathcal{M}$ it must do so with total degree zero. One manifestation of this observation is that if we take $\mathcal{M}$ to be a surface of revolution rotated around the z-axis and if we take $\mathbf{H}^e$ to be constant and vertical, then we can construct two-vortex critical points of the energy, with zeros lying at the north and south poles, cf. Proposition 4.1. This construction and related properties of the solutions are discussed in Section 4.

Continuing with the assumption that $\mathcal{M}$ is a surface of revolution and $\mathbf{H}^e$ is constant and vertical, we conclude this article in Section 5 with a determination of the asymptotic value of the first critical field H_{c1} in the 'extreme type-II regime' where $\kappa \gg 1$ and $|\mathbf{H}^e| \sim \ln \kappa$. For the case of an infinite superconducting cylinder of constant cross-section, the authors of [21] carry out such an investigation and determine the critical coefficient of $\ln \kappa$, characterizing it in terms of a solution to a certain auxiliary problem related to the London equation. (See also [22] for much more detailed information about H_{c1} in this setting.) For the planar problem arising as a thin film limit, the authors of [5, 6] determine this critical coefficient in terms of a different auxiliary problem. In the present manifold setting, we first obtain an asymptotic upper bound on H_{c1} through a construction. Then we obtain a lower bound that matches the upper bound so that, somewhat remarkably, the first critical field is simply given by

$$H_{c1} = \left(\frac{4\pi}{\text{area of } \mathcal{M}} \right) \ln \kappa$$

in the large κ regime, cf. Theorem 5.1. Combining this with the Γ-convergence and compactness results of Section 3, we can establish a similar result for the 3d Ginzburg-Landau energy posed on Ω_ε when ε is small, cf. Theorem 5.3. To our knowledge, this is one of the first calculations of the first critical field for Ginzburg-Landau in a three-dimensional setting, preceded by the determination of a candidate for H_{c1} for a solid ball in $\mathbb{R}^3$ in [1].

Our proof of the lower bound requires us to adapt and when necessary substantially adjust the technology on energy concentration on balls developed in [14] and [21]. Aside from its use in analyzing H_{c1} in the present paper and more recently in [3], we expect that this extension of energy ball concentration results to the manifold setting will prove useful in other investigations involving Ginzburg-Landau on manifolds.

We view this article as a 'first shot' at the rigorous analysis of Ginzburg-Landau on a manifold in the presence of an applied field. Further results related to the large κ regime for surfaces of revolution will be presented in [4]. Certainly in the future it would be illuminating to study the more subtle roles that geometry and topology can play by considering more general manifolds, to investigate the critical field H_{c3} associated with loss of stability of the normal state, and of course, to look at the dynamics of Ginzburg-Landau vortices on manifolds.

2. Basic definitions, notation and conventions

We will use X to denote a point in $\mathbb{R}^3$. For any two-dimensional manifold S, we will use $\mathcal{H}^2_S$ to denote two-dimensional Hausdorff measure restricted to S. We let $\mathcal{M}$ denote a two-dimensional, C^2, orientable and compact manifold in $\mathbb{R}^3$, with or without boundary, and use x or p to denote a point on $\mathcal{M}$. We write $\nu(x)$ for a unit normal to the manifold at a given point $x \in \mathcal{M}$, in particular it denotes the outer unit normal in the case where $\mathcal{M}$ has no boundary. In light of the assumed regularity of $\mathcal{M}$, we can assert the existence of a value $\varepsilon_0 > 0$ such that the map $T_\varepsilon : \mathcal{M} \times (0,1) \to \mathbb{R}^3$ given by

$$X = T_\varepsilon(x,t) := x + \varepsilon t \nu(x), \tag{2.1}$$

is smoothly invertible for all $\varepsilon \in (0, \varepsilon_0)$. We shall assume the superconductor occupies a thin neighborhood of $\mathcal{M}$ given by

$$\Omega_\varepsilon := \{X \in \mathbb{R}^3 : \ X = x + \varepsilon t \nu(x) \text{ for } x \in \mathcal{M}, \ t \in (0,1)\},$$

for $\varepsilon < \varepsilon_0$.

Our object of study will be the Ginzburg-Landau functional

$$G_{\varepsilon,\kappa}(\Psi, \mathbf{A}) = \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \left(|(\nabla - i\mathbf{A})\Psi|^2 + \frac{\kappa^2}{2} \left(|\Psi|^2 - 1 \right)^2 \right) dX + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} |\nabla \times \mathbf{A} - \mathbf{H}^e|^2 \, dX. \tag{2.2}$$

Here the external magnetic field $\mathbf{H}^e : \mathbb{R}^3 \to \mathbb{R}^3$ is taken to be a given, smooth divergence-free vector field. The constant $\kappa > 0$ is the Ginzburg-Landau parameter and the scaling by $1/\varepsilon$ is chosen to keep energies at $\mathcal{O}(1)$. As is natural, we take $G_{\varepsilon,\kappa}$ to be defined for $\Psi \in H^1(\Omega_\varepsilon; \mathbb{C})$. Regarding the domain of definition of the potential $\mathbf{A}$, we introduce $\mathcal{H}$ as the closure of

$$\{\mathbf{A} \in C^\infty(\mathbb{R}^3; \mathbb{R}^3) : \ \mathbf{A} \text{ compactly supported}\}$$

with respect to the norm $\|\nabla \mathbf{A}\|_{L^2(\mathbb{R}^3;\mathbb{R}^9)} = \left(\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \, dx\right)^{1/2}$. Then we set $\mathcal{H}_0 = \{\mathbf{A} \in \mathcal{H} : \operatorname{div} \mathbf{A} = 0\}$. We note here that for $\mathbf{A} \in \mathcal{H}$ one has

$$\|\mathbf{A}\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} \leq C \left(\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \, dX\right)^{1/2}, \tag{2.3}$$

and for $\mathbf{A} \in \mathcal{H}_0$ one has

$$\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \, dX = \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 \, dX. \tag{2.4}$$

We also choose a magnetic potential $\mathbf{A}^e$ corresponding to the given external magnetic field $\mathbf{H}^e$ to be any vector field satisfying the requirements

$$\nabla \times \mathbf{A}^e = \mathbf{H}^e \quad \text{and } \operatorname{div} \mathbf{A}^e = 0 \quad \text{in } \mathbb{R}^3. \tag{2.5}$$

These conditions determine $\mathbf{A}^e$ up to the gradient of a harmonic function. With these definitions in place, we then view $G_{\varepsilon,\kappa}$ as being defined for all $\mathbf{A}$ such that $\mathbf{A} - \mathbf{A}^e \in \mathcal{H}_0$.

Notice that through the invertible map T_ε, we can associate to each $\Psi \in H^1(\Omega_\varepsilon; \mathbb{C})$ a function $\psi \in H^1(\mathcal{M} \times (0,1); \mathbb{C})$ via the formula $\psi(x,t) = \Psi(T_\varepsilon(x,t))$. Then, denoting by $\nabla_{\mathcal{M}}\psi$ the tangential gradient of ψ relative to $\mathcal{M}$, one directly calculates from (2.1) that

$$\nabla \Psi(X) = \nabla_{\mathcal{M}}\psi(x,t) + \frac{1}{\varepsilon}\psi_t(x,t)\nu(x) + \varepsilon \chi_\varepsilon(x,t), \tag{2.6}$$

where $\chi_\varepsilon(x,t)$ is a vector field tangent to $\mathcal{M}$ satisfying the bound $|\chi_\varepsilon| \leq C |\nabla \Psi|$ for some constant C depending only on $\mathcal{M}$.

For any two-dimensional manifold S, we will use $\mathcal{H}^2_S$ to denote two-dimensional Hausdorf measure restricted to S. Now for any $t \in (0,1)$ and $\varepsilon \in (0, \varepsilon_0)$, we introduce the manifold

$$\mathcal{M}_{\varepsilon,t} := \{x + \varepsilon t \nu(x) : \ x \in \mathcal{M}\}, \tag{2.7}$$

and observe that (2.2) can be written as

$$G_{\varepsilon,\kappa}(\Psi,\mathbf{A}) = \int_0^1 \int_{\mathcal{M}_{\varepsilon,t}} \left(|(\nabla - i\mathbf{A})\Psi|^2 + \frac{\kappa^2}{2}\left(|\Psi|^2-1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}_{\varepsilon,t}}(X)dt$$
$$+\frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\nabla\times\mathbf{A}-\mathbf{H}^e|^2\, dX. \tag{2.8}$$

Making the change of variables $X = T_\varepsilon(x,t)$, using (2.6) and noting from (2.1) that

$$d\mathcal{H}^2_{\mathcal{M}_{\varepsilon,t}}(X) = (1+\mathcal{O}(\varepsilon))d\mathcal{H}^2_{\mathcal{M}}(x), \tag{2.9}$$

we can assert that

$$G_{\varepsilon,\kappa}(\Psi,\mathbf{A}) = \mathcal{G}_{\varepsilon,\kappa}(\psi,\mathbf{A}) + \mathcal{O}(\varepsilon)G_{\varepsilon,\kappa}(\Psi,\mathbf{A}), \tag{2.10}$$

where

$$\mathcal{G}_{\varepsilon,\kappa}(\psi,\mathbf{A}) :=$$
$$\int_0^1\int_{\mathcal{M}} \left(\left|(\nabla_{\mathcal{M}} - i\mathbf{A}^\tau)\psi\right|^2 + \left|(\frac{1}{\varepsilon}\nu\partial_t - i\mathbf{A}^\nu)\psi\right|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\right) d\mathcal{H}^2_{\mathcal{M}}(x)\,dt$$
$$+\frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\nabla\times\mathbf{A}-\mathbf{H}^e|^2\, dX. \tag{2.11}$$

Here we have introduced notation for the normal and tangential components of any potential $\mathbf{A}$ near $\mathcal{M}$, namely

$$\mathbf{A}^\nu(x,t) := A^\nu\nu \text{ where } A^\nu := \mathbf{A}(T_\varepsilon(x,t))\cdot\nu(x) \quad\text{and}\quad \mathbf{A}^\tau(x,t) := \mathbf{A}(T_\varepsilon(x,t)) - \mathbf{A}^\nu(x,t).$$

In particular, we will write $(\mathbf{A}^e)^\nu$ and $(\mathbf{A}^e)^\tau$ for the normal and tangential components of the applied potential $\mathbf{A}^e$ near $\mathcal{M}$ viewed as functions of x and t.

Note that the $\mathcal{O}(\varepsilon)$ discrepancy between $G_{\varepsilon,\kappa}(\Psi,\mathbf{A})$ and $\mathcal{G}_{\varepsilon,\kappa}(\psi,\mathbf{A})$ one sees in (2.10) arises from the $\mathcal{O}(\varepsilon)$ error terms appearing in (2.6) and (2.9).

3. Γ-convergence

In this section we will identify the Γ-limit of the 3d Ginzburg-Landau energy $G_{\varepsilon,\kappa}$ as $\varepsilon\to 0$. Following this, we will establish a compactness result for energy-bounded sequences and a stronger compactness result for sequences of minimizers. Then we will conclude with a general result showing that for both the Γ-limit and for $G_{\varepsilon,\kappa}$, there is a first critical field H_{c1} such that for applied fields below this value, a minimizer will have no vortices. This last result will be significantly sharpened later in the paper when we take the field to be vertical and constant and take the limiting manifold $\mathcal{M}$ to be a surface of revolution.

Before stating our proposed Γ-limit of $\{G_{\varepsilon,\kappa}\}$, we must first introduce the topology of the convergence. To this end, given $(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \subset H^1(\Omega_\varepsilon;\mathbb{C}) \times (\{\mathbf{A}^e\} + \mathcal{H}_0)$ and $(\psi, \mathbf{A}) \in H^1(\mathcal{M} \times (0,1);\mathbb{C}) \times (\{\mathbf{A}^e\} + \mathcal{H}_0)$ we will write $(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \xrightarrow{Y} (\psi, \mathbf{A})$ provided

$$\psi^\varepsilon \rightharpoonup \psi \text{ weakly in } H^1(\mathcal{M} \times (0,1);\mathbb{C}) \quad \text{and} \quad \mathbf{A}^\varepsilon - \mathbf{A} \to 0 \text{ strongly in } \mathcal{H}_0, \tag{3.1}$$

where $\psi^\varepsilon = \Psi^\varepsilon \circ T_\varepsilon$. (See Remark 3.3 below.)

Then for $(\psi, \mathbf{A}) \in H^1(\mathcal{M};\mathbb{C}) \times (\{\mathbf{A}^e\} + \mathcal{H}_0)$ we define

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}} \left(|(\nabla_{\mathcal{M}} - i(\mathbf{A}^e)^\tau)\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x) \tag{3.2}$$

and for $(\psi, \mathbf{A}) \in H^1(\mathcal{M} \times (0,1);\mathbb{C}) \times (\{\mathbf{A}^e\} + \mathcal{H}_0)$ we define

$$G_{\mathcal{M},\kappa}(\psi, \mathbf{A}) = \begin{cases} \mathcal{G}_{\mathcal{M},\kappa}(\psi) & \text{if } \psi_t = 0 \; a.e. \text{ in } \mathcal{M} \times (0,1),\; \mathbf{A} = \mathbf{A}^e, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

We point out that in (3.3) we have made the obvious identification between elements ψ of $H^1(\mathcal{M} \times (0,1);\mathbb{C})$ satisfying the condition $\psi_t = 0$ a.e. and elements of $H^1(\mathcal{M};\mathbb{C})$.

Theorem 3.1. *The sequence of functionals $G_{\varepsilon,\kappa}$ Γ-converges as $\varepsilon \to 0$ to $G_{\mathcal{M},\kappa}$ in the Y-topology.*

The definition of Γ-convergence consists of conditions (3.5) and (3.9) below. Before beginning the proof we first present a needed lemma.

Lemma 3.2. *Let $\{(\Psi^\varepsilon, \mathbf{A}^\varepsilon)\}$ be any sequence in $H^1(\Omega_\varepsilon;\mathbb{C}) \times (\{\mathbf{A}^e\} + \mathcal{H}_0)$ satisfying a uniform energy bound $G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) < C_0$ for some $C_0 > 0$. Then there exists a constant $C > 0$ independent of ε such that*

$$\left(\int_0^1 \int_{\mathcal{M}} |\mathbf{A}^\varepsilon \circ T_\varepsilon - \mathbf{A}^e \circ T_\varepsilon|^6 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt \right)^{1/6} \leq C\varepsilon^{1/3}. \tag{3.4}$$

Proof of Lemma 3.2. Through the change of variables $s = \varepsilon t$, property (2.9) and Hölder's inequality we find that

$$\left(\int_0^1 \int_{\mathcal{M}} |\mathbf{A}^\varepsilon \circ T_\varepsilon - \mathbf{A}^e \circ T_\varepsilon|^6 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/6} =$$
$$\frac{1}{\varepsilon^{1/6}} \left(\int_0^\varepsilon \int_{\mathcal{M}} |\mathbf{A}^\varepsilon(x + s\nu(x)) - \mathbf{A}^e(x + s\nu(x))|^6 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, ds\right)^{1/6} \leq$$
$$\frac{C}{\varepsilon^{1/6}} \left(\int_{\Omega_\varepsilon} |\mathbf{A}^\varepsilon - \mathbf{A}^e|^6 \, dX\right)^{1/6}.$$

Then using (2.3), (2.4) and the uniform energy bound we can estimate

$$\|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^6(\Omega_\varepsilon;\mathbb{R}^3)} \leq \|\mathbf{A}^\varepsilon - \mathbf{A}^e\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} \leq C \left(\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}^\varepsilon - \mathbf{H}^e|^2 \, dx\right)^{1/2} \leq C\varepsilon^{1/2}.$$

Combining these two inequalities, we have established (3.4). □

Proof of Theorem 3.1.

Lower-semi-continuity. We begin with a proof of the assertion:

$$\text{(3.5)} \qquad \text{Whenever } (\Psi^\varepsilon, \mathbf{A}^\varepsilon) \xrightarrow{Y} (\psi, \mathbf{A}) \text{ one has } \liminf_{\varepsilon \to 0} G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \geq G_{\mathcal{M},\kappa}(\psi, \mathbf{A}).$$

Let us assume that

$$\liminf_{\varepsilon \to 0} G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) < +\infty,$$

as otherwise there is nothing to prove. But in this case, one sees from (2.2) that necessarily, $\mathbf{A} = \mathbf{A}^e$ and from (2.10) it will suffice to show that

$$\liminf_{\varepsilon \to 0} \mathcal{G}_{\varepsilon,\kappa}(\psi^\varepsilon, \mathbf{A}^\varepsilon) \geq G_{\mathcal{M},\kappa}(\psi, \mathbf{A}^e) \tag{3.6}$$

provided

$$\mathcal{G}_{\varepsilon,\kappa}(\psi^\varepsilon, \mathbf{A}^\varepsilon) \leq C_0 \quad \text{for some} \quad C_0 > 0. \tag{3.7}$$

We begin the verification of (3.6) by noting that Lemma 3.2, (3.7) and Hölder's inequality imply

$$\int_0^1 \int_{\mathcal{M}} |(\mathbf{A}^\varepsilon)^\nu \psi^\varepsilon|^2 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt \le$$
$$C\left(\int_0^1 \int_{\mathcal{M}} |(\mathbf{A}^\varepsilon)^\nu - (\mathbf{A}^e)^\nu|^4 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} \left(\int_0^1 \int_{\mathcal{M}} |\psi^\varepsilon|^4 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} +$$
$$C\left(\int_0^1 \int_{\mathcal{M}} |(\mathbf{A}^e)^\nu|^4 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} \left(\int_0^1 \int_{\mathcal{M}} |\psi^\varepsilon|^4 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} < C$$

since $\mathbf{A}^e$ is locally bounded in L^∞. (Here and throughout, C denotes a positive constant independent of ε.) Consequently, we see that

$$\int_0^1 \int_{\mathcal{M}} |\psi^\varepsilon_t|^2 \, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt < C\varepsilon^2.$$

Thus, $\psi_t = 0$ a.e.in $\mathcal{M}\times(0,1)$ and we may restrict our attention to the case where $G_{\mathcal{M},\kappa}(\psi,\mathbf{A}) = \mathcal{G}_{\mathcal{M},\kappa}(\psi)$. But through (2.11) we find that

$$\liminf_{\varepsilon\to 0} \mathcal{G}_{\varepsilon,\kappa}(\psi^\varepsilon,\mathbf{A}^\varepsilon) \geq \liminf_{\varepsilon\to 0}\int_0^1\int_{\mathcal{M}}\left(\left|\big(\nabla_{\mathcal{M}} - i(\mathbf{A}^\varepsilon)^\tau\big)\psi^\varepsilon\right|^2 + \frac{\kappa^2}{2}\left(|\psi^\varepsilon|^2-1\right)^2\right) d\mathcal{H}^2_{\mathcal{M}}(x)\, dt$$

$$= \liminf_{\varepsilon\to 0}\int_0^1\int_{\mathcal{M}}\Bigg(|\nabla_{\mathcal{M}}\psi^\varepsilon|^2 + i\big((\psi^\varepsilon)^*\nabla_{\mathcal{M}}\psi^\varepsilon - \psi^\varepsilon\nabla_{\mathcal{M}}(\psi^\varepsilon)^*\big)\cdot(\mathbf{A}^\varepsilon)^\tau + |(\mathbf{A}^\varepsilon)^\tau\psi^\varepsilon|^2$$

$$+\frac{\kappa^2}{2}\left(|\psi^\varepsilon|^2-1\right)^2\Bigg)\, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt$$

$$\geq \liminf_{\varepsilon\to 0}\int_0^1\int_{\mathcal{M}}\Bigg(|\nabla_{\mathcal{M}}\psi^\varepsilon|^2 + i\big((\psi^\varepsilon)^*\nabla_{\mathcal{M}}\psi^\varepsilon - \psi^\varepsilon\nabla_{\mathcal{M}}(\psi^\varepsilon)^*\big)\cdot(\mathbf{A}^e)^\tau + |(\mathbf{A}^e)^\tau\psi^\varepsilon|^2$$

$$+\frac{\kappa^2}{2}\left(|\psi^\varepsilon|^2-1\right)^2\Bigg)\, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt$$

$$+\liminf_{\varepsilon\to 0}\int_0^1\int_{\mathcal{M}}\left(\left(|(\mathbf{A}^\varepsilon)^\tau\psi^\varepsilon|^2 - |(\mathbf{A}^e)^\tau\psi^\varepsilon|^2\right)\right) d\mathcal{H}^2_{\mathcal{M}}(x)\, dt$$

$$+\liminf_{\varepsilon\to 0}\int_0^1\int_{\mathcal{M}}\left(i\big((\psi^\varepsilon)^*\nabla_{\mathcal{M}}\psi^\varepsilon - \psi^\varepsilon\nabla_{\mathcal{M}}(\psi^\varepsilon)^*\big)\cdot\left((\mathbf{A}^\varepsilon)^\tau - (\mathbf{A}^e)^\tau\right)\right) d\mathcal{H}^2_{\mathcal{M}}(x)\, dt$$

$$= \quad \liminf_{\varepsilon\to 0}\ I \quad + \quad \liminf_{\varepsilon\to 0}\ II \quad + \quad \liminf_{\varepsilon\to 0}\ III. \tag{3.8}$$

In light of the weak H^1-convergence of $\{\psi^\varepsilon\}$, we know (up to subsequences) that $\psi^\varepsilon \to \psi$ strongly in L^4. Hence, we may pass to the limit in I to obtain $\liminf_{\varepsilon\to 0}\ I \geq G_{\mathcal{M},\kappa}(\psi,\mathbf{A}^e)$. Thus, (3.6) will follow if we can demonstrate that $\lim_{\varepsilon\to 0} II = 0$ and $\lim_{\varepsilon\to 0} III = 0$.

Turning to the limit of II, we estimate using (3.4) and the boundedness of $\mathbf{A}^e$ along with Hölder's inequality again that

$$\begin{aligned}
&\left|\int_0^1 \int_{\mathcal{M}} \Big(\left(|(\mathbf{A}^\varepsilon)^\tau \psi^\varepsilon|^2 - |(\mathbf{A}^e)^\tau \psi^\varepsilon|^2\right) \Big)\, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right| = \\
&\left|\int_0^1 \int_{\mathcal{M}} \Big(\left(|(\mathbf{A}^\varepsilon)^\tau| - |(\mathbf{A}^e)^\tau|\right) |\psi^\varepsilon|^2 \left(|(\mathbf{A}^\varepsilon)^\tau| + |(\mathbf{A}^e)^\tau|\right) \Big)\, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right| \leq \\
&C\left(\int_0^1 \int_{\mathcal{M}} |(\mathbf{A}^\varepsilon)^\tau - (\mathbf{A}^e)^\tau|^2\, |\psi^\varepsilon|^4\; d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} \cdot \\
&\left(\int_0^1 \int_{\mathcal{M}} \left(|(\mathbf{A}^\varepsilon)^\tau|^2 + |(\mathbf{A}^e)^\tau|^2\right)\; d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} \leq \\
&C\left(\int_0^1 \int_{\mathcal{M}} |(\mathbf{A}^\varepsilon)^\tau - (\mathbf{A}^e)^\tau|^2\, |\psi^\varepsilon|^4\; d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} \cdot \\
&\left(\int_0^1 \int_{\mathcal{M}} \left(|(\mathbf{A}^\varepsilon)^\tau - (\mathbf{A}^e)^\tau|^2 + |(\mathbf{A}^e)^\tau|^2\right)\; d\mathcal{H}^2_{\mathcal{M}}(x)\, dt\right)^{1/2} \\
&\leq C\, \|\mathbf{A}^\varepsilon \circ T_\varepsilon - \mathbf{A}^e \circ T_\varepsilon\|_{L^6(\mathcal{M}\times(0,1))}\, \|\psi^\varepsilon\|^2_{L^6(\mathcal{M}\times(0,1))} < C\varepsilon^{1/3}
\end{aligned}$$

since

$$\|\psi^\varepsilon\|_{L^6(\mathcal{M}\times(0,1))} \leq C\, \|\psi^\varepsilon\|_{H^1(\mathcal{M}\times(0,1))} < C$$

in light of the weak H^1 convergence of $\{\psi^\varepsilon\}$. Hence, $\lim_{\varepsilon\to 0} II = 0$.

The estimate of III in (3.8) is handled similarly:

$$\begin{aligned}
&\int_0^1 \int_{\mathcal{M}} \left| i\big((\psi^\varepsilon)^* \nabla_{\mathcal{M}} \psi^\varepsilon - \psi^\varepsilon \nabla_{\mathcal{M}} (\psi^\varepsilon)^*\big) \cdot \left((\mathbf{A}^\varepsilon)^\tau - (\mathbf{A}^e)^\tau\right)\right|\, d\mathcal{H}^2_{\mathcal{M}}(x)\, dt \leq \\
&C\, \|\nabla\psi^\varepsilon\|_{L^2(\mathcal{M}\times(0,1))}\, \|\psi^\varepsilon\|_{L^4(\mathcal{M}\times(0,1))}\, \|(\mathbf{A}^\varepsilon)^\tau - (\mathbf{A}^e)^\tau\|_{L^4(\mathcal{M}\times(0,1))} \leq C\varepsilon^{1/3}.
\end{aligned}$$

Construction of recovery sequence. Given any $\psi \in H^1(\mathcal{M}\times(0,1))$ and $\mathbf{A} \in \{\mathbf{A}^e\} + \mathcal{H}_0$, our goal here is to construct a sequence $\{(\Psi^\varepsilon, \mathbf{A}^\varepsilon)\}$ satisfying the conditions

$$(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \xrightarrow{Y} (\psi, \mathbf{A}) \quad \text{and} \quad \lim_{\varepsilon\to 0} G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) = G_{\mathcal{M},\kappa}(\psi, \mathbf{A}). \tag{3.9}$$

If either $\mathbf{A} \neq \mathbf{A}^e$ or if $\psi_t \neq 0$ on a set of positive measure in $\mathcal{M}\times(0,1)$, then the trivial choice $(\Psi^\varepsilon, \mathbf{A}^\varepsilon) = (\psi(T_\varepsilon^{-1}), \mathbf{A})$ for all $\varepsilon > 0$ serves as a recovery sequence since then $G_{\mathcal{M},\kappa}(\psi, \mathbf{A}) = \infty$ and one easily verifies that

$$\lim_{\varepsilon\to 0} G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) = \infty.$$

Thus, we proceed under the assumption that $\mathbf{A} = \mathbf{A}^e$ and $\psi_t = 0$ a.e. so that $\psi = \psi(x)$ only and $G_{\mathcal{M},\kappa}(\psi, \mathbf{A}) = \mathcal{G}_{\mathcal{M},\kappa}(\psi)$. In this case, for $\mathbf{A}^\varepsilon$ we choose simply $\mathbf{A}^\varepsilon = \mathbf{A}^e$ for all $\varepsilon > 0$. For the construction of Ψ^ε we will utilize the solution $\psi^\varepsilon : \mathcal{M} \times (0,1) \to \mathbb{C}$ to:

$$\begin{cases} \partial_t \psi^\varepsilon(x,t) = i\varepsilon\big(\mathbf{A}^e(x + \varepsilon t\nu(x)) \cdot \nu(x)\big)\, \psi^\varepsilon(x,t) & \text{for } x \in \mathcal{M},\ t \in (0,1), \\ \psi^\varepsilon(x,0) = \psi(x), \end{cases} \tag{3.10}$$

an ordinary differential equation in t in which x plays the role of a parameter. In other words

$$\psi^\varepsilon(x,t) = \psi(x) e^{i\varepsilon \int_0^t (\mathbf{A}^e(x+\varepsilon s\nu(x))\cdot\nu(x))\, ds}. \tag{3.11}$$

Then we define $\Psi^\varepsilon := \psi^\varepsilon \circ T_\varepsilon^{-1}$. Direct calculation reveals that

$$|\psi^\varepsilon - \psi|^2 + |\nabla_{\mathcal{M}}\psi^\varepsilon - \nabla_{\mathcal{M}}\psi|^2 + |\psi_t^\varepsilon|^2 \leq C\varepsilon\, |\psi|^2 . \tag{3.12}$$

Consequently, $\psi^\varepsilon \to \psi$ strongly in $H^1(\mathcal{M} \times (0,1))$ and so in particular, Ψ^ε converges in the Y-topology introduced in (3.1).

Substituting this choice for $(\Psi^\varepsilon, \mathbf{A}^\varepsilon)$ into $G_{\varepsilon,\kappa}$ and using (2.10), (2.11) and (3.12) we find

$$\begin{aligned} \lim_{\varepsilon\to 0} G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) &= \int_0^1 \int_{\mathcal{M}} \left(\big|(\nabla_{\mathcal{M}} - i\mathbf{A}^\tau)\psi^\varepsilon\big|^2 + \frac{\kappa^2}{2}\left(|\psi^\varepsilon|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x)\, dt \\ &= \int_0^1 \int_{\mathcal{M}} \left(\big|(\nabla_{\mathcal{M}} - i\mathbf{A}^\tau)\psi\big|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x)\, dt \\ &= \int_{\mathcal{M}} \left(\big|(\nabla_{\mathcal{M}} - i\mathbf{A}^\tau)\psi\big|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x) = \mathcal{G}_{\mathcal{M},\kappa}(\psi) = G_{\mathcal{M},\kappa}(\psi, \mathbf{A}). \end{aligned}$$

Note that the second term in (2.11) is completely eliminated in light of (3.10). □

Remark 3.3. An examination of the recovery sequence construction reveals that we could have strengthened the topology of the Γ-convergence to be *strong* H^1-convergence on $\mathcal{M}\times(0,1)$ rather than weak. We have stated the result for weak convergence so as to work in a topology for which we also have compactness of energy-bounded sequences.

We turn now to the issue of compactness.

Proposition 3.4. *Given any sequence* $\{(\Psi^\varepsilon, \mathbf{A}^\varepsilon)\} \subset H^1(\Omega_\varepsilon; \mathbb{C}) \times \big(\{\mathbf{A}^e\} + \mathcal{H}_0\big)$, *satisfying a uniform energy bound*

$$G_{\varepsilon,\kappa}(\Psi^\varepsilon, \mathbf{A}^\varepsilon) \leq C,$$

there exists a function $\psi \in H^1(\mathcal{M};\mathbb{C})$ *such that after passing to a subsequence one has*

$$\psi_\varepsilon := \Psi^\varepsilon(T_\varepsilon) \rightharpoonup \psi \text{ weakly in } H^1(\mathcal{M}\times(0,1);\mathbb{C}) \quad \text{and} \quad (\psi_\varepsilon)_t \to 0 \text{ strongly in } L^2(\mathcal{M}\times(0,1);\mathbb{C}) \tag{3.13}$$

while

$$\mathbf{A}^\varepsilon - \mathbf{A}^e \to 0 \quad \text{strongly in } \mathcal{H}_0. \tag{3.14}$$

Proof. The uniform energy bound $G_{\varepsilon,\kappa}(\Psi^\varepsilon,\mathbf{A}^\varepsilon) \leq C$ immediately implies (3.14). Then through (2.3), (2.10), (2.11) and Lemma 3.2 we see that both ψ_ε and $\mathbf{A}^\varepsilon \circ T_\varepsilon$ are uniformly bounded in $L^4(\mathcal{M}\times(0,1))$. Hence the uniform bound on $\mathcal{G}_{\varepsilon,\kappa}(\psi_\varepsilon,\mathbf{A}^\varepsilon)$ and use of Hölder's inequality also yield

$$\int_0^1\int_{\mathcal{M}}\left(|\nabla_{\mathcal{M}}\psi_\varepsilon|^2 + \frac{1}{\varepsilon^2}|(\psi_\varepsilon)_t|^2\right)d\mathcal{H}^2_{\mathcal{M}}(x)\,dt \leq C\left(1+\int_0^1\int_{\mathcal{M}}\left(|\psi_\varepsilon|^2\,|\mathbf{A}^\varepsilon\circ T_\varepsilon|^2\right)d\mathcal{H}^2_{\mathcal{M}}(x)\,dt\right) \leq C,$$

and (3.13) follows. □

The topology of convergence can naturally be strengthened if one considers not just energy-bounded sequences but instead sequences of minimizers of $G_{\varepsilon,\kappa}$.

Proposition 3.5. *Fix any* $\kappa > 0$. *For any* $\varepsilon > 0$, *let* $\Psi_{\varepsilon,\kappa} : \Omega_\varepsilon \to \mathbb{C}$ *and* $\mathbf{A}_{\varepsilon,\kappa} : \mathbb{R}^3 \to \mathbb{R}^3$ *denote a minimizing pair for* $G_{\varepsilon,\kappa}$ *with* $\psi_{\varepsilon,\kappa} : \mathcal{M}\times(0,1) \to \mathbb{C}$ *associated with* $\Psi_{\varepsilon,\kappa}$ *via* $\psi_{\varepsilon,\kappa}(x,t) := \Psi_{\varepsilon,\kappa}(x+t\varepsilon\nu(x))$. *Then there exists a subsequence* $\{\varepsilon_j\} \to 0$ *and a minimizer* ψ_κ *of* $\mathcal{G}_{\mathcal{M},\kappa}$ *such that* $\psi_{\varepsilon_j,\kappa} \to \psi_\kappa$ *in* $C^{0,\alpha}\big(\mathcal{M}\times(0,1)\big)$ *for any positive* $\alpha < \frac{1}{2}$.

Proof. We begin by invoking the Γ-convergence of Theorem 3.1 and compactness of Proposition 3.4 to conclude that a minimizer $(\Psi_{\varepsilon,\kappa},\mathbf{A}_{\varepsilon,\kappa})$ of $G_{\varepsilon,\kappa}$ satisfies

$$\psi_{\varepsilon,\kappa} \rightharpoonup \psi_\kappa \text{ weakly in } H^1(\mathcal{M}\times(0,1);\mathbb{C}), \quad (\psi_{\varepsilon,\kappa})_t \to 0 \text{ strongly in } L^2(\mathcal{M}\times(0,1);\mathbb{C}) \tag{3.15}$$

where ψ_κ minimizes $\mathcal{G}_{\mathcal{M},\kappa}$.

Our goal is to upgrade the topology of the convergence to $C^{0,\alpha}$ and to this end, we note that a minimizing pair $(\Psi_{\varepsilon,\kappa}, \mathbf{A}_{\varepsilon,\kappa})$ satisfies the elliptic Euler-Lagrange system

$$(\nabla - i\mathbf{A}_{\varepsilon,\kappa})^2 \Psi_{\varepsilon,\kappa} = \kappa^2 \left(|\Psi_{\varepsilon,\kappa}|^2 - 1\right) \Psi_{\varepsilon,\kappa} \quad \text{in } \Omega_\varepsilon, \tag{3.16}$$

$$\nabla \times \nabla \times \mathbf{A}_{\varepsilon,\kappa} = \begin{cases} \frac{i}{2}\left(\Psi_{\varepsilon,\kappa}\nabla\Psi^*_{\varepsilon,\kappa} - \Psi^*_{\varepsilon,\kappa}\nabla\Psi_{\varepsilon,\kappa}\right) - |\Psi_{\varepsilon,\kappa}|^2 \mathbf{A}_{\varepsilon,\kappa} & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \mathbb{R}^3 \setminus \Omega_\varepsilon, \end{cases} \tag{3.17}$$

along with the conditions

$$(\nabla - i\mathbf{A}_{\varepsilon,\kappa}) \Psi_{\varepsilon,\kappa} \cdot \nu_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

and $\nabla \times \mathbf{A}_{\varepsilon,\kappa} - \mathbf{H}^e \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. Also, the comparison $G_{\varepsilon,\kappa}(\Psi_{\varepsilon,\kappa}, \mathbf{A}_{\varepsilon,\kappa}) \leq G_{\varepsilon,\kappa}(1, \mathbf{A}^e)$ yields the bounds

$$\kappa^2 \int_{\Omega_\varepsilon} \left(|\Psi_{\varepsilon,\kappa}|^2 - 1\right)^2 dX \leq C \|\mathbf{H}^e\|^2_{L^2(B;\mathbb{R}^3)} \varepsilon, \quad \int_{\Omega_\varepsilon} |\nabla\Psi_{\varepsilon,\kappa}|^2 \, dX \leq C \|\mathbf{H}^e\|^2_{L^2(B;\mathbb{R}^3)} \varepsilon, \tag{3.18}$$

along with

$$\left(\int_{\Omega_\varepsilon} |\mathbf{A}_{\varepsilon,\kappa} - \mathbf{A}^e|^6 \, dX\right)^{1/6} \leq C \|\mathbf{H}^e\|^2_{L^2(B;\mathbb{R}^3)} \sqrt{\varepsilon}, \tag{3.19}$$

after an appeal to (2.3) and (2.4). Here we can take $B \subset \mathbb{R}^3$ to be any ball containing, say $\{X : \text{ dist } (\mathrm{X}, \mathcal{M}) < 1\}$.

Combining (3.18) and (3.19) with the elementary inequality $|\Psi_{\varepsilon,\kappa}| \leq 1$ that follows from the maximum principle, we find through (3.16) that

$$\|\Delta\Psi_{\varepsilon,\kappa}\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon},$$

where here and below in the proof, $C > 0$ denotes a constant that is independent of ε, but which could depend on κ. Hence, by standard elliptic regularity, cf. [13], one has

$$\|\Delta\Psi_{\varepsilon,\kappa}\|_{H^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}. \tag{3.20}$$

In light of the assumed smoothness of the manifold $\mathcal{M}$, it then follows that the map $\Psi_{\varepsilon,\kappa} \circ T_1$ satisfies a similar H^2-bound on the set $\mathcal{M} \times (0, \varepsilon)$, where $T_1 = T_1(x, s) := x + s\nu(x)$, cf. (2.1). Substituting $s = \varepsilon t$ then results in an ε-independent bound

$$\|\psi_{\varepsilon,\kappa}\|_{H^2(\mathcal{M}\times(0,1))} \leq C. \tag{3.21}$$

The result follows from the compact imbedding of H^2 in $C^{0,\alpha}$. □

Having obtained the Γ-limit $\mathcal{G}_{\mathcal{M},\kappa}$ of the original Ginzburg-Landau energy, we turn now to the goal of showing that for sufficiently small applied fields, minimizers do not vanish. In other words, we want to establish a general statement that there is a threshold in field strength that must be crossed before vortices appear in an energy minimizer. We begin with the Γ-limit.

Proposition 3.6. *For any smooth, divergence-free* $\mathbf{H}^e : \mathbb{R}^3 \to \mathbb{R}^3$*, let* $\{\mathbf{H}^e_\delta\}_{\delta>0}$ *denote the family of applied fields given by* $\mathbf{H}^e_\delta = \delta\mathbf{H}^e$*. Also denote by* $\mathcal{G}^\delta_{\mathcal{M},\kappa}$ *the corresponding energy*

$$\mathcal{G}^\delta_{\mathcal{M},\kappa}(\psi) := \int_{\mathcal{M}} \left(|(\nabla_{\mathcal{M}} - i(\mathbf{A}^e_\delta)^\tau)\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x),$$

where $\mathbf{A}^e_\delta = \delta\mathbf{A}^e$ *and* $\mathbf{A}^e$ *is any divergence-free magnetic potential satisfying* $\nabla\times\mathbf{A}^e = \mathbf{H}^e$*. Then there is a positive value* δ_0 *such that for* $\delta \in [0,\delta_0]$*, any global minimizer of* $\mathcal{G}^\delta_{\mathcal{M},\kappa}$ *does not vanish.*

Proof. It is simple to check that as $\delta \to 0$, $\mathcal{G}^\delta_{\mathcal{M},\kappa}$ Γ-converges in the weak H^1 topology to

$$\mathcal{G}^0_{\mathcal{M},\kappa}(\psi) := \int_{\mathcal{M}} \left(|\nabla_{\mathcal{M}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x).$$

The lower-semi-continuity follows easily from the convex dependence of the energy on $\nabla_{\mathcal{M}}\psi$ and the trivial sequence $\psi^\delta = \psi$ suffices as a recovery sequence. Compactness in the weak H^1-topology is equally simple.

As a result of this Γ-convergence and the uniform energy bound $\mathcal{G}^\delta_{\mathcal{M},\kappa}(\psi^\delta) \le \mathcal{G}^\delta_{\mathcal{M},\kappa}(1) \le C\delta^2$ for any sequence $\{\psi^\delta\}$ of minimizers of $\mathcal{G}^\delta_{\mathcal{M},\kappa}$, one has weak convergence in H^1, up to subsequences, of $\{\psi^\delta\}$ to a minimizer ψ^0 of $\mathcal{G}^0_{\mathcal{M},\kappa}$. Clearly, $\psi^0 = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. In addition to the uniform H^1-bound on $\{\psi^\delta\}$, one can use the Euler-Lagrange equation,

$$(\nabla_{\mathcal{M}} - i(\mathbf{A}^e_\delta)^\tau)^2\psi^\delta = \kappa^2(\left|\psi^\delta\right|^2 - 1)\psi^\delta$$

to see via standard elliptic regularity that

$$\left\|\Delta_{\mathcal{M}}\psi^\delta\right\|_{L^2(\mathcal{M};\mathbb{C})} \le C\left\|\psi^\delta\right\|_{H^1(\mathcal{M};\mathbb{C})} < C.$$

Hence the sequence of minimizers is uniformly bounded in H^2 and converges uniformly, up to subsequences, to some $e^{i\alpha}$. In particular, the full sequence $\{|\psi^\delta|\}$ must converge uniformly on $\mathcal{M}$ to 1. The result follows.

□

We conclude this section with a similar result for the 3d energy:

Proposition 3.7. *Fix any $\kappa > 0$. For $\mathbf{H}^e_\delta$ defined as in Proposition 3.6, let*

$$G^\delta_{\varepsilon,\kappa}(\Psi,\mathbf{A}) := \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left(|(\nabla - i\mathbf{A})\Psi|^2 + \frac{\kappa^2}{2}\left(|\Psi|^2-1\right)^2\right)dX + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\nabla\times\mathbf{A}-\mathbf{H}^e_\delta|^2\,dX$$

and let $(\Psi^\delta_{\varepsilon,\kappa},\mathbf{A}^\delta_{\varepsilon,\kappa})$ denote any minimizing pair. Then for any positive $\delta < \delta_0$, with δ_0 coming from Proposition 3.6, there exists a value $\varepsilon_0 = \varepsilon_0(\delta)$, such that for all positive $\varepsilon < \varepsilon_0$, $\Psi^\delta_{\varepsilon,\kappa}$ does not vanish.

Proof. Fixing any $\delta < \delta_0$, we may apply the Γ-convergence result Theorem 3.1 and the compactness result Proposition 3.5 to the sequence of functionals $G^\delta_{\varepsilon,\kappa}$ and sequence of minimizers $(\Psi^\delta_{\varepsilon,\kappa},\mathbf{A}^\delta_{\varepsilon,\kappa})$ as $\varepsilon \to 0$. Suppose then, by way of contradiction, that along a subsequence $\{\varepsilon_j\} \to 0$, the functions $\Psi^\delta_{\varepsilon_j,\kappa}$ had a zero for each ε_j somewhere in Ω_{ε_j}. Then $\psi^\delta_{\varepsilon_j,\kappa}$ would have to vanish somewhere on $\mathcal{M}\times(0,1)$ for each ε_j. Consequently, the uniform convergence of $\psi^\delta_{\varepsilon_j,\kappa}$ to a minimizer ψ^δ of $\mathcal{G}^\delta_{\mathcal{M},\kappa}$ guaranteed by Proposition 3.5 and the compactness of $\mathcal{M}$ would imply that ψ^δ must vanish somewhere on $\mathcal{M}$. This contradicts Proposition 3.6. □

4. Critical points of $G_{\mathcal{M},\kappa}$ for surfaces of revolution

Next we undertake the construction of non-trivial critical points of the Γ-limit (3.2) under the assumption that the surface $\mathcal{M}$ is a smooth compact surface of revolution without boundary which is a topological sphere. Our primary goal here is to initiate an investigation of the first critical field, H_{c1}, for such a surface, namely the value of an external field above which the global minimizer of $\mathcal{G}_{\mathcal{M},\kappa}$ must have vortices. Throughout this section and for the remainder of the paper, we will take $\mathbf{H}^e$ to be of the form $h(0,0,1)$ for some non-negative value of h.

We will describe the surface using standard spherical coordinates, that is, using θ to denote the standard azimuth angle and ϕ to denote the zenith angle. Then let $u : [0, \pi] \to \mathbb{R}$ and $v : [0, \pi] \to \mathbb{R}$ be C^1 functions related by the condition

$$v(\phi) = \cot\phi\, u(\phi) \quad \text{for } 0 < \phi < \pi \tag{4.1}$$

with

$$u(0) = 0 = u(\pi), \quad v(0) > 0,\ v(\pi) > 0 \quad \text{and} \quad v'(0) = 0 = v'(\pi). \tag{4.2}$$

We will realize the manifold as the surface obtained through revolution about the z-axis of the curve

$$(u(\phi), 0, v(\phi)) \quad \text{for } 0 \le \phi \le \pi \tag{4.3}$$

lying in the xz-plane. Let us further assume that this parametrized curve is regular in the sense that

$$\gamma(\phi) := \sqrt{u'(\phi)^2 + v'(\phi)^2} \ge \gamma_0 \quad \text{for } \phi \in [0, \pi] \tag{4.4}$$

for some positive constant γ_0. Note that necessarily,

$$u(\phi) = l\phi + o(\phi) \quad \text{for some positive constant } l \tag{4.5}$$

near $\phi = 0$ with a similar expansion holding near $\phi = \pi$.

This leads to the simply connected, parametrized surface of revolution $\check{\mathcal{M}}$ defined by

$$\check{\mathcal{M}} := \{(u(\phi)\cos\theta, u(\phi)\sin\theta, v(\phi)) : \phi \in [0, \pi], \theta \in [0, 2\pi]\}. \tag{4.6}$$

If we denote by $\hat{e}_\theta$ and $\hat{e}_\phi$ the unit vectors in the θ and ϕ directions respectively, then for any function $\psi : \check{\mathcal{M}} \to \mathbb{C}$ we have

$$\nabla_{\check{\mathcal{M}}}\psi = \frac{1}{\gamma(\phi)}\psi_\phi \hat{e}_\phi + \frac{1}{u(\phi)}\psi_\theta \hat{e}_\theta. \tag{4.7}$$

Also, for the area element, we note that

$$d\mathcal{H}^2_{\check{\mathcal{M}}} = u(\phi)\,\gamma(\phi)\,d\phi\,d\theta. \tag{4.8}$$

Regarding the magnetic potential associated with the field $h(0,0,1)$, it will be convenient to choose $\mathbf{A}^e = \frac{h}{2}(-X_2, X_1, 0)$ so that on $\check{\mathcal{M}}$ we have

$$\mathbf{A}^e = (\mathbf{A}^e)^\tau = \frac{hu(\phi)}{2}\hat{e}_\theta. \tag{4.9}$$

Thus, the functional $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ given by (3.2) takes the form

$$\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi) := \int_0^\pi \int_0^{2\pi} \left\{ \frac{1}{\gamma^2} |\psi_\phi|^2 + \left| \frac{1}{u}\psi_\theta - i\frac{hu}{2}\psi \right|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 \right\} u\,\gamma\, d\theta\, d\phi. \tag{4.10}$$

.

Exploiting the symmetry of the problem and with the intuition gained by observing the computations of [10, 11], we seek a two-vortex critical point with vortices at the north and south pole in the form $\psi(\theta,\phi) = f(\phi)e^{i\theta}$ for some function $f : [0,\pi] \to \mathbb{R}$ vanishing at the endpoints. Plugging this ansatz into (4.10), we are left with the task of finding a non-trivial critical point f of the functional

$$E_{h,\kappa}(f) := \int_0^\pi \left\{ \frac{f'^2}{\gamma^2} + \left(\frac{1}{u} - \frac{hu}{2} \right)^2 f^2 + \frac{\kappa^2}{2}(f^2 - 1)^2 \right\} \gamma\, u\, d\phi. \tag{4.11}$$

We note that $f \equiv 0$ is always a critical point of $E_{h,\kappa}$, as it is of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ for that matter.

Then let $\mathcal{R}_h$ denote the Rayleigh quotient

$$\mathcal{R}_h(f) := \frac{\int_0^\pi \left\{ \frac{f'^2}{\gamma^2} + \left(\frac{1}{u} - \frac{hu}{2}\right)^2 f^2 \right\} \gamma\, u\, d\phi}{\int_0^\pi f^2\, \gamma\, u\, d\phi}$$

with associated first eigenvalue

$$\lambda_1(h) := \inf_f \mathcal{R}_h(f).$$

Now we can prove:

Proposition 4.1. *For any $h \geq 0$ and for $\kappa^2 > \lambda_1(h)$, there exists a non-trivial global minimizer, $f_{h,\kappa}$, of $E_{h,\kappa}(f)$ within the admissible set*

$$\mathcal{A} := L_2((0,\pi)) \cap \left\{ f \in H^1_{\mathrm{loc}}((0,\pi)) : \int_0^\pi \frac{1}{\gamma} f'^2 u(\phi)\, d\phi < \infty\,, \int_0^\pi \frac{f^2}{u(\phi)}\gamma\, d\phi < \infty. \right\}.$$

The minimizer $f_{h,\kappa}$ is continuous on $[0,\pi]$ and satisfies the problem

$$-\frac{1}{\gamma^2}f''_{h,\kappa}-\frac{1}{\gamma u}\left(\frac{u}{\gamma}\right)'f'_{h,\kappa}+\left(\frac{1}{u}-\frac{hu}{2}\right)^2 f_{h,\kappa}+\kappa^2(f^3_{h,\kappa}-f_{h,\kappa})=0 \quad on \quad 0<\phi<\pi, \tag{4.12}$$

with $f_{h,\kappa}(0)=f_{h,\kappa}(\pi)=0$. Hence, $\psi=f_{h,\kappa}(\phi)e^{i\theta}$ represents a non-trivial critical point of $\mathcal{G}_{\check{\mathcal{M}}}$ having vortices at the north and south poles. On the other hand, when $\kappa^2<\lambda_1(h)$, the unique global minimizer of $E_{h,\kappa}$ is $\psi\equiv 0$.

Remark 4.2. *Consider the special case where $\check{\mathcal{M}}$ is the unit sphere S^2 so that $u=\sin\phi$, $v=\cos\phi$ and $\gamma=1$, and suppose there is no applied field ($h=0$). Then the eigenvalue problem associated with the Rayleigh quotient $\mathcal{R}_0$ is*

$$-(\sin\phi)f''-(\cos\phi)f'+\frac{1}{\sin\phi}f=\lambda_1(0)(\sin\phi)f$$

and one can explicitly compute that $\lambda_1(0)=2$ with corresponding first eigenfunction $\sin\phi$. Hence, our theorem guarantees the existence of a non-trivial 2-vortex critical point on S^2 in the absence of any applied field provided $\kappa^2>2$.

Remark 4.3. We conjecture that for h in some interval $(\underline{h},\overline{h})$ with $0<\underline{h}<\overline{h}<\infty$, and for $\kappa^2>\lambda(h)$, the two-vortex critical point $f_{h,\kappa}(\phi)e^{i\theta}$ is in fact a local minimizer of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ and in a perhaps smaller subinterval it is the global minimizer. While we have estimates on the Morse index of these critical points, so far we have not established these conjectured stability properties in general. However, in the asymptotic regime $\kappa\gg 1$ and $h=h(\kappa)\sim\ln\kappa$, this stability question seems to be more tractable, [4].

Remark 4.4. By similar methods, one can also construct critical points with higher degree zeros at the north and south poles by plugging the ansatz $\psi(\theta,\phi)=f_n(\phi)e^{in\theta}$ for any integer $n\geq 2$ into $\mathcal{G}_{\check{\mathcal{M}},\kappa}$. This leads to the minimization problem

$$\inf_f E^{(n)}_{h,\kappa}(f) \quad \text{where } E^{(n)}_{h,\kappa}(f):=\int_0^\pi\left\{\frac{f'^2}{\gamma^2}+\left(\frac{n}{u}-\frac{hu}{2}\right)^2f^2+\frac{\kappa^2}{2}(f^2-1)^2\right\}\gamma\, u\, d\phi. \tag{4.13}$$

The direct method will again provide a minimizer which, as before, will be non-trivial provided that κ^2 exceeds the first eigenvalue $\lambda_1^{(n)}(h)$ associated with the Rayleigh quotient

$$\mathcal{R}_h^{(n)}(f) := \frac{\int_0^\pi \left\{ \frac{f'^2}{\gamma^2} + \left(\frac{n}{u} - \frac{hu}{2}\right)^2 f^2 \right\} \gamma\, u\, d\phi}{\int_0^\pi f^2\, \gamma\, u\, d\phi}.$$

Proof. We will obtain $f_{h,\kappa}$ via the direct method. To this end, fix any $h \geq 0$ and $\kappa > 0$ and let $\{f_j\} \subset \mathcal{A}$ denote a minimizing sequence for $E_{h,\kappa}$. Then $E_{h,\kappa}(f_j) \leq E_{h,\kappa}(0) = \frac{\kappa^2}{2}\mathcal{H}^2(\check{\mathcal{M}}) =: c_0$ so we immediately conclude that for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\|f_j\|_{H^1((\delta,\pi-\delta))} \leq C_\delta \quad \text{for all } j.$$

Hence we find via a diagonalization argument that after passing to a subsequence (with subsequential notation suppressed), one has

$$f_j \rightharpoonup f_{h,\kappa} \quad \text{in } H^1_{\text{loc}}((0,\pi))$$

for some function $f_{h,\kappa} \in H^1_{\text{loc}}((0,\pi))$. Of course, this implies that the convergence is also locally uniform and that $f_{h,\kappa}$ is continuous on $(0,\pi)$. Then by the lower semi-continuity of the Sobolev norm, we find for each $\delta > 0$ that

$$\begin{aligned} c_0 \geq \liminf_{j\to\infty} E_{h,\kappa}(f_j) \geq \int_\delta^{\pi-\delta} \Bigg\{ \frac{(f_j')^2}{\gamma^2} \\ + \left(\frac{1}{u} - \frac{h}{2}u\right)^2 f_j^2 + \frac{\kappa^2}{2}(f_j^2-1)^2 \Bigg\} \gamma\, u\, d\phi \\ \geq \int_\delta^{\pi-\delta} \Bigg\{ \frac{(f_{h,\kappa}')^2}{\gamma^2} + \left(\frac{1}{u} - \frac{h}{2}u\right)^2 f_{h,\kappa}^2 + \\ \frac{\kappa^2}{2}(f_{h,\kappa}^2-1)^2 \Bigg\} \gamma\, u\, d\phi \end{aligned}$$

Letting $\delta \to 0$, we conclude that $E_{h,\kappa}(f_{h,\kappa}) \leq c_0$ so that indeed $f_{h,\kappa} \in \mathcal{A}$ and that $f_{h,\kappa}$ minimizes $E_{h,\kappa}$. It remains to argue that $f_{h,\kappa}$ is continuous on $[0,\pi]$ and vanishes at the endpoints. To this end, we calculate that for any ϕ_1 and ϕ_2 in $(0,\pi)$:

$$\left|f_{h,\kappa}^2(\phi_2) - f_{h,\kappa}^2(\phi_1)\right| = \left|2\int_{\phi_1}^{\phi_2} f_{h,\kappa} f'_{h,\kappa}\, d\phi\right|$$
$$= 2\left|\int_{\phi_1}^{\phi_2} f_{h,\kappa}\left(\frac{\gamma}{u}\right)^{\frac{1}{2}} f'_{h,\kappa}\left(\frac{u}{\gamma}\right)^{\frac{1}{2}} d\phi\right|$$
$$\leq 2\left(\int_{\phi_1}^{\phi_2} \frac{f_{h,\kappa}^2\gamma}{u}\, d\phi\right)^{\frac{1}{2}}\left(\int_{\phi_1}^{\phi_2} \frac{f'^2_{h,\kappa}u}{\gamma}\, d\phi\right)^{\frac{1}{2}}.$$

Since $E_{h,\kappa}(f_{h,\kappa}) \leq c_0$ implies in particular that the last two integrals both approach zero as $|\phi_1 - \phi_2| \to 0$, we see that $f_{h,\kappa}^2$ is uniformly continuous on $(0,\pi)$. Thus, $f_{h,\kappa}$ naturally has a continuous extension to $[0,\pi]$ with $f_{h,\kappa}(0) = 0 = f_{h,\kappa}(\pi)$.

It remains to argue that the minimizer $f_{h,\kappa}$ is non-trivial provided $\kappa^2 > \lambda_1(h)$ while it is identically zero if $\kappa^2 < \lambda_1(h)$. To see this, consider the second variation of $E_{h,\kappa}$ taken about the critical point $f \equiv 0$:

$$\delta^2 E_{h,\kappa}(0; f) = \int_0^\pi \left\{\frac{f'^2}{\gamma^2} + \left(\frac{1}{u} - \frac{hu}{2}\right)^2 f^2 - \kappa^2 f^2\right\}\gamma\, u\, d\phi.$$

If $\kappa^2 > \lambda_1(h)$, then letting f_1 denote the first eigenfunction corresponding to $\lambda_1(h)$ we find

$$\delta^2 E_{h,\kappa}(0; f_1) = \int_0^\pi \left\{(\lambda_1(h) - \kappa^2) f_1^2\right\}\gamma\, u\, d\phi < 0,$$

and so 0 is unstable. Hence, the global minimizer $f_{h,\kappa}$ is non-trivial.

On the other hand, if $\kappa^2 < \lambda_1(h)$ then we calculate that for any nontrivial $f \in \mathcal{A}$ one has

$$E_{h,\kappa}(f) - E_{h,\kappa}(0) = \left(\mathcal{R}_h(f) - \kappa^2\right)\int_0^\pi f^2\gamma u\, d\phi + \frac{\kappa^2}{2}\int_0^\pi f^4\gamma u\, d\phi$$
$$\geq \left(\lambda_1(h) - \kappa^2\right)\int_0^\pi f^2\gamma u\, d\phi + \frac{\kappa^2}{2}\int_0^\pi f^4\gamma u\, d\phi > 0,$$

and so 0 is the unique global minimizer and $f_{h,\kappa}$ constructed via the direct method is trivial.

□

Proposition 4.1 establishes the existence of a two-vortex critical point on a surface of revolution when $\kappa^2 > \lambda_1(h)$ by realizing its modulus as the minimizer of $E_{h,\kappa}$. However, this leaves open the possibility that other non-trivial critical points of $E_{h,\kappa}$ exist when $\kappa^2 < \lambda(h)$ that are either unstable or at least not globally minimizing. We partially address this in the following proposition which relates the possibility of having a non-trivial two-vortex critical point of degree n to the geometry of the curve (4.3).

Proposition 4.5. *Let n be any positive integer and suppose*

$$u_{\max}^2 < \frac{2n}{h} \quad \left(\text{so that} \quad \frac{n}{u_{\max}} - \frac{h}{2}u_{\max} > 0 \right) \quad \text{where} \quad u_{\max} := \max_{[0,\pi]} u(\phi).$$

If

$$\kappa^2 \leq \left(\frac{n}{u_{\max}} - \frac{h}{2}u_{\max} \right)^2,$$

then the only critical point of $E_h^{(n)}$ (cf. (4.13)) is $\psi \equiv 0$.

Proof. Suppose that f is a critical point of $E_h^{(n)}$ so that f satisfies (4.12) with the coefficient of $f_{h,\kappa}$ in the third term replaced more generally by $\left(\frac{n}{u} - \frac{hu}{2}\right)^2$. Suppose also that f is non-trivial (and if $f(\phi) < 0$ for all $\phi \in (0,\pi)$ then replace f by $-f$) so that

$$0 < a := \max_{[0,\pi]} f(\phi) = f(\phi_0) \quad \text{for some } \phi_0 \in (0,\pi).$$

Then at $\phi = \phi_0$, (4.12) yields

$$\left(\frac{n}{u(\phi_0)} - \frac{h}{2}u(\phi_0) \right)^2 a + \kappa^2(a^3 - a) \leq 0.$$

Hence,

$$0 < a^2 \leq 1 - \frac{1}{\kappa^2}\left(\frac{n}{u(\phi_0)} - \frac{h}{2}u(\phi_0) \right)^2 \leq 1 - \frac{1}{\kappa^2}\left(\frac{n}{u_{\max}} - \frac{h}{2}u_{\max} \right)^2,$$

so that

$$\kappa^2 > \left(\frac{n}{u_{\max}} - \frac{h}{2}u_{\max} \right)^2.$$

□

5. Estimates for H_{c1} on manifolds in the large κ regime

In this section we continue the analysis of the Γ-limit $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ given by (4.10) where again we take $\mathcal{M} = \check{\mathcal{M}}$ to be a surface of revolution described by (4.1)-(4.6). However, in order to apply a version of the results from [12], we will further assume in this section that the functions u and v, and hence the manifold $\check{\mathcal{M}}$ are analytic. As in the previous section, for given external magnetic field $\mathbf{H}^e = h(0,0,1)$, we seek information about H_{c1}, the smallest value of h above which the global minimizer must possess vortices. What distinguishes this section from the previous one is that here we consider a so-called extreme type-II superconductor by taking κ to be large and studying the asymptotic regime where $h = h(\kappa)$ obeys

$$\lim_{\kappa\to\infty} \frac{h(\kappa)}{\ln\kappa} = C_0 \tag{5.1}$$

for some non-negative constant C_0.

To describe the results, for any $\phi \in (0,\pi)$, we introduce notation for the circle $C_\phi \subset \check{\mathcal{M}}$ via

$$C_\phi := \{\big(u(\phi)\cos\theta, u(\phi)\sin\theta, v(\phi)\big) :\ 0 \le \theta \le 2\pi\}. \tag{5.2}$$

Then we will establish:

Theorem 5.1. *(Part I) If the magnitude of the external field satisfies* (5.1) *with* $C_0 > \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$, *then there exists a value* κ_0 *such that for all* $\kappa \ge \kappa_0$, *any global minimizer* ψ_κ *of* $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ *satisfies the conditions*

$$\psi_\kappa \neq 0 \text{ everywhere on } C_\phi \quad \text{and} \quad \deg(\psi_\kappa, C_\phi) \neq 0 \quad \text{for some } \phi \in (0,\pi) \tag{5.3}$$

where $\deg(\psi_\kappa, C_\phi)$ *is simply the winding number of* ψ_κ *restricted to* C_ϕ. *In particular,* ψ_κ *has at least two vortices of nonzero degree.*
(Part II) If, instead the external field satisfies (5.1) *with* $C_0 < \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$, *then there exists a value* κ_0 *such that for all* $\kappa \ge \kappa_0$, *any global minimizer of* $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ *does not vanish.*

Remark 5.2. This relatively simply asymptotic formula for H_{c1} will not hold in general if one relaxes the assumption that the manifold is a surface of revolution. Note that for surfaces with this symmetry, the magnetic potential $\mathbf{A}^e$ is purely tangential, allowing for substantial saving in energy through the term $\Lambda(\mathbf{A}^e, \psi)$, cf. (5.5). It is not hard to construct examples of manifolds where $\mathbf{H}^e$ is mostly tangential to the surface, and hence there would exist a potential $\mathbf{A}^e$ for which $\Lambda(\mathbf{A}^e, \psi)$ vanishes on most of the surface for any ψ. Based on these ideas, we conjecture that $4\pi/\mathcal{H}^2(\mathcal{M})$ is the smallest possible coefficient of $\ln\kappa$ for H_{c1} that one can obtain.

As an application of the Γ-convergence result Theorem 3.1, we will show in this section that the value $\frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$ also serves as an asymptotic value for H_{c1} for the 3d Ginzburg-Landau energy $G_{\varepsilon,\kappa}$ when ε is sufficiently small. More precisely, we will show:

Theorem 5.3. *(i) Assume* $C_0 > \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$ *in condition* (5.1)*. Fix any value* $\kappa \geq \kappa_0$ *where* κ_0 *is the value arising in Theorem 5.1 and for any* $\varepsilon > 0$*, let* Ψ_ε *denote a minimizer of* $G_{\varepsilon,\kappa}$*. Then there exists a value* $\varepsilon_0 = \varepsilon_0(\kappa)$ *such that for all positive* $\varepsilon < \varepsilon_0$ *there exists a circle* $C_{\phi_\varepsilon} \subset \check{\mathcal{M}}$ *as defined in* (5.2) *satisfying the condition*

$$\Psi_\varepsilon \neq 0 \text{ everywhere on } C_{\phi_\varepsilon,t}, \quad \deg(\Psi_\varepsilon, C_{\phi_\varepsilon,t}) \neq 0 \quad \text{for all} \quad t \in (0,1) \tag{5.4}$$

where

$$C_{\phi_\varepsilon,t} := \{x + \varepsilon t\nu(x) : \ x \in C_{\phi_\varepsilon}\} \subset \check{\mathcal{M}}_{\varepsilon,t},$$

cf. (2.7)*. Hence, in particular,* Ψ_ε *vanishes at least twice on each manifold* $\check{\mathcal{M}}_{\varepsilon,t}$*, for* $0 < t < 1$*.*

(ii) Assume $C_0 < \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$ *in condition* (5.1)*. Fix any value* $\kappa \geq \kappa_0$ *where* κ_0 *is the value arising in Theorem 5.1 and for any* $\varepsilon > 0$*, let* $\Psi_{\varepsilon,\kappa}$ *denote a minimizer of* $G_{\varepsilon,\kappa}$*. Then there exists a value* $\varepsilon_0 = \varepsilon_0(\kappa)$ *such that for all positive* $\varepsilon < \varepsilon_0$*,* $\Psi_{\varepsilon,\kappa}$ *does not vanish in* Ω_ε*.*

The proof of this theorem will be given at the end of the section in Propositions 5.10 and 5.11. In fact, in Proposition 5.10 we will prove a bit more than is stated above in item (i).

The section is split into three parts. In the first, we obtain an asymptotic upper bound by analyzing the case where $C_0 > \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$. In the second part, we obtain an asymptotic lower bound corresponding to the reverse inequality. Together, these two results constitute a proof of Theorem 5.1. In the last part we prove Theorem 5.3.

5.1. **Asymptotic upper bound on H_{c1} (Part I of Theorem 5.1).** We begin by studying the case where the applied field is sufficiently large to force the presence of vortices in minimizers of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$.

The proof of Part I of Theorem 5.1 relies on the following lemma.

Lemma 5.4. *For any* $\mathbf{A} \in H^1(\check{\mathcal{M}};\mathbb{R}^3)$ *and any* $\psi \in H^1(\check{\mathcal{M}};\mathbb{C})$ *define*

$$\Lambda(\mathbf{A},\psi) := i\int_{\check{\mathcal{M}}} \mathbf{A}^\tau \cdot (\psi\nabla_{\check{\mathcal{M}}}\psi^* - \psi^*\nabla_{\check{\mathcal{M}}}\psi)\, d\mathcal{H}^2_{\check{\mathcal{M}}}, \tag{5.5}$$

where as before, $\mathbf{A}^\tau := \mathbf{A} - \big(\mathbf{A}\cdot\nu\big)\nu$*. Given any* $\psi \in C^1(\check{\mathcal{M}};\mathbb{C})$ *satisfying* $\mathcal{G}_{\mathcal{M},\kappa}(\psi) \le \mathcal{G}_{\mathcal{M},\kappa}(1)$*, suppose that* $\mathcal{H}^1(S) = \pi$ *where*

$$S := \{\phi \in (0,\pi) : \psi \neq 0 \text{ everywhere on } C_\phi \quad \text{and} \quad \deg(\psi, C_\phi) = 0\}. \tag{5.6}$$

Then

$$|\Lambda(\mathbf{A}^e,\psi)| \le C_u \frac{h(\kappa)^3}{\kappa} = \mathcal{O}\bigg(\frac{(\ln\kappa)^3}{\kappa}\bigg), \tag{5.7}$$

where $C_u > 0$ *is a constant depending only on* u *given in* (4.3)*.*

Proof. For each $\phi \in S$, we may write $\psi(\theta,\phi) = f(\theta,\phi)e^{i\chi(\theta,\phi)}$ for $0 \le \theta \le 2\pi$ where f and χ are smooth real functions that are 2π-periodic in θ. Then we see from (4.7)–(4.9) that

$$\begin{aligned}\Lambda(\mathbf{A}^e,\psi) \quad &= h(\kappa)\int_S\int_0^{2\pi} f^2\,\frac{\partial\chi}{\partial\theta}\,u\,\gamma\,d\theta\,d\phi\\ &= h(\kappa)\int_S\int_0^{2\pi} \big(f^2-1\big)\,\frac{\partial\chi}{\partial\theta}\,u\,\gamma\,d\theta\,d\phi,\end{aligned}$$

since the periodicity of χ implies that

$$\int_S\bigg(\int_0^{2\pi}\frac{\partial\chi}{\partial\theta}(\theta,\phi)\,d\theta\bigg)\,u(\phi)\,\gamma(\phi)\,d\phi = 0. \tag{5.8}$$

Then we use the assumption

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) \le \mathcal{G}_{\mathcal{M},\kappa}(1) = \bigg(\frac{h(\kappa)}{2}\bigg)^2\|u\|^2_{L^2(\check{\mathcal{M}})}, \tag{5.9}$$

along with an application of Hölder's inequality to conclude that

$$\begin{aligned}|\Lambda(\mathbf{A}^e,\psi)| \quad &= \bigg|h(\kappa)\int_S\int_0^{2\pi}\big(f^2-1\big)\,\frac{\partial\chi}{\partial\theta}\,u\,\gamma\,d\theta\,d\phi\bigg|\\ &\le u_{\max}h(\kappa)\bigg|\int_S\int_0^{2\pi}\sqrt{u\gamma}\,\big(f^2-1\big)\,\frac{1}{u}\frac{\partial\chi}{\partial\theta}\,\sqrt{u\gamma}d\theta\,d\phi\bigg|\\ &\le u_{\max}h(\kappa)\bigg(\int_{\check{\mathcal{M}}}(|\psi|^2-1)^2\,d\mathcal{H}^2_{\check{\mathcal{M}}}\bigg)^{\frac12}\bigg(\int_{\check{\mathcal{M}}}|\nabla_{\check{\mathcal{M}}}\psi|^2\,d\mathcal{H}^2_{\check{\mathcal{M}}}\bigg)^{\frac12}\\ &\le C_u\frac{h(\kappa)^3}{\kappa}.\end{aligned}$$

Invoking (5.1), we complete the proof. ☐

Proof of Part I of Theorem 5.1.

We begin by constructing a two-vortex function

$$\tilde{\psi}_\kappa(\theta,\phi) = f_\kappa(\phi)e^{i\theta}, \tag{5.10}$$

where

$$f_\kappa(\phi) := \begin{cases} 0, & \phi \in [0, \frac{1}{2\kappa}) \\ 2\kappa\left(\phi - \frac{1}{2\kappa}\right), & \phi \in [\frac{1}{2\kappa}, \frac{1}{\kappa}) \\ 1, & \phi \in [\frac{1}{\kappa}, \frac{\pi}{2}] \\ f_\kappa(\pi - \phi), & \phi \in [\frac{\pi}{2}, \pi]. \end{cases} \tag{5.11}$$

Then taking ψ_κ to be a global minimizer of $\mathcal{G}_{\mathcal{M},\kappa}$, it follows that

$$\int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau \psi_\kappa|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} - \Lambda(\mathbf{A}^e, \psi_\kappa) \le \mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) \le \mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\psi}_\kappa), \tag{5.12}$$

so that we will have a lower bound for the quantity $\Lambda(\mathbf{A}^e, \psi_\kappa)$ once we compute $\mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\psi}_k)$. The proof hinges on pitting this lower bound against the upper bound provided by Lemma 5.4 satisfied by a vortex-free minimizer.

We proceed to estimate each of the terms arising in $\mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\psi}_\kappa)$, cf. (4.11). In the estimates below, all terms denoted by $\mathcal{O}(1)$ refer to terms bounded by a constant that may depend on $\check{\mathcal{M}}$ through the functions u or v but which are independent of κ.

First of all, it is easy to check using (4.5) that

$$\int_0^\pi \left\{\frac{1}{\gamma^2} f_\kappa'^2 + \kappa^2(f_\kappa^2 - 1)^2\right\} u\,\gamma\, d\phi = \mathcal{O}(1). \tag{5.13}$$

Then we estimate that

$$\int_0^\pi \left(\frac{h(\kappa)}{2}\right)^2 f_\kappa^2 u^3 \gamma \, d\phi \le \int_0^\pi \left(\frac{h(\kappa)}{2}\right)^2 u^3\gamma\, d\phi = \left(\frac{h(\kappa)}{2}\right)^2 \frac{1}{2\pi} \|u\|^2_{L^2(\check{\mathcal{M}})}. \tag{5.14}$$

Now from (4.2) and (4.5), one sees that

$$\frac{\gamma}{u} = \frac{1}{\phi} + \mathcal{O}(1) \quad \text{for } \phi \text{ near } 0$$

with a similar estimate holding near $\phi = \pi$, from which it follows that

$$\int_0^\pi \frac{1}{u} f_\kappa^2 \gamma\, d\phi \le 2\ln\kappa + \mathcal{O}(1). \tag{5.15}$$

Finally, one checks that

$$
\begin{aligned}
h(\kappa)\int_0^\pi f_\kappa^2 u\gamma\, d\phi &= h(\kappa)\int_0^\pi u\gamma\, d\phi - h(\kappa)\int_0^\pi (1-f_\kappa^2)u\gamma\, d\phi \\
&= h(\kappa)\frac{\mathcal{H}^2(\check{\mathcal{M}})}{2\pi} - \mathcal{O}\big(\frac{h(\kappa)}{\kappa^2}\big) \\
&\geq h(\kappa)\frac{\mathcal{H}^2(\check{\mathcal{M}})}{2\pi} - o(1), \qquad (5.16)
\end{aligned}
$$

where we have invoked (5.1) in the last inequality. (We seek a lower bound for this term rather than an upper bound since it appears with a minus sign in the energy.) From (5.13)-(5.16), we conclude that

$$
\mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\psi}_\kappa) \leq \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} + 4\pi \ln \kappa - \mathcal{H}^2(\check{\mathcal{M}})h(\kappa) + \mathcal{O}(1). \qquad (5.17)
$$

Returning to (5.12), we have found that

$$
\begin{aligned}
\Lambda(\mathbf{A}^e, \psi_\kappa) &\geq \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau \psi_\kappa|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} - \mathcal{G}_{\check{\mathcal{M}},\kappa}(\tilde{\psi}_\kappa) \\
&\geq \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau \psi_\kappa|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} - \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - 4\pi \ln \kappa + h(\kappa)\mathcal{H}^2(\check{\mathcal{M}}) - \mathcal{O}(1). \qquad (5.18)
\end{aligned}
$$

Then since a minimizer ψ_κ in particular satisfies the estimate (5.9), we know that

$$
\int_{\check{\mathcal{M}}} (|\psi_\kappa|^2 - 1)^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} \leq \frac{\|u\|^2_{L^2(\check{\mathcal{M}})}}{4} \left(\frac{h(\kappa)}{\kappa}\right)^2.
$$

Thus, by (4.9) and Hölder's inequality,

$$
\begin{aligned}
&\int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau \psi_\kappa|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} = \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} - \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau|^2 (1 - |\psi_\kappa|^2) \, d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&= \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - \left(\frac{h(\kappa)}{2}\right)^2 \int_{\check{\mathcal{M}}} |u|^2 (1 - |\psi_\kappa|^2) \, d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&\geq \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^4(\check{\mathcal{M}})} \left(\int_{\check{\mathcal{M}}} (|\psi_\kappa|^2 - 1)^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}}\right)^{\frac{1}{2}} \\
&\geq \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - \mathcal{O}\left(\frac{h(\kappa)^3}{\kappa}\right). \qquad (5.19)
\end{aligned}
$$

Consequently, first (5.18) and then (5.1) yield the lower bound

$$\begin{aligned} \Lambda(\mathbf{A}^e,\psi_\kappa) &\geq -4\pi\ln\kappa + \mathcal{H}^2(\check{\mathcal{M}})h(\kappa) - \mathcal{O}(1) - \mathcal{O}\Big(\frac{h(\kappa)^3}{\kappa}\Big) \\ &\geq \big(\mathcal{H}^2(\check{\mathcal{M}}) - \frac{4\pi}{C_0}\big)h(\kappa) - \mathcal{O}(1) =: \mathcal{R}(\kappa). \end{aligned} \tag{5.20}$$

Since we are assuming that $C_0 > \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$, we see that the coefficient of $h(\kappa)$ in the first term of the lower bound in (5.20) is positive. Since any minimizer ψ_κ is smooth, Lemma 5.4 applies and by choosing κ_0 large enough so that, say,

$$C_u\frac{h(\kappa)^3}{\kappa} < \frac{1}{2} \quad \text{while} \quad \mathcal{R}(\kappa) > 1 \quad \text{provided } \kappa \geq \kappa_0, \tag{5.21}$$

we conclude that $\mathcal{H}^1\big((0,\pi)\setminus S\big) > 0$ for $\kappa \geq \kappa_0$ where S is given by (5.6).

Finally, in light of the simple connectivity and analyticity of $\check{\mathcal{M}}$ and consequent analyticity of ψ_κ, we may apply the main result of [12] to conclude that the zeros of ψ_κ are isolated. (The result of [12] is established for planar domains but inspection of the proof reveals that all of the analysis carries over to the case of a 2d, analytic, simply connected manifold.) Since $\check{\mathcal{M}}$ is compact without boundary, the zero set is therefore finite and so in particular, there must exist a set of ϕ-values of positive measure for which $\psi_\kappa \neq 0$ on C_ϕ and $\deg(\psi_\kappa, C_\phi) \neq 0$. Focusing on any one such ϕ, the result follows since C_ϕ divides $\check{\mathcal{M}}$ into two disjoint components, each one necessarily containing at least one vortex. □

5.2. **Asymptotic lower bound on H_{c1} (Proof of Theorem 5.1 Part II).** In this subsection our goal is to obtain an asymptotic lower bound on the size of the first critical field. We will see that it coincides with the upper bound obtained in the previous subsection. In order to achieve this we will need to adapt results from [14] and [21] regarding energy concentration on balls to the manifold setting.

We will denote by $\exp_p$ the exponential map at p, cf. [7]. It is well known that for r small enough, $\exp_p$ provides a local diffeomorphism from $T_p\check{\mathcal{M}}$ onto its image in $\check{\mathcal{M}}$. This radius will be denoted henceforth by $\tilde{r}$, which can be chosen independently of the point p in virtue of the compactness of the surface. That is, we fix $\tilde{r}$ to be any positive value below the injectivity radius of the surface. We then define a *pseudo-ball* to be the diffeomorphic image of a Euclidean ball under the exponential map, i.e. $\hat{B}(p,r) := \exp_p[B(0,r)]$ for $B(0,r) \subset T_p\check{\mathcal{M}}$.

We begin with a proposition whose content is essentially Proposition 3.2 of [21].

Proposition 5.5. *Let ψ be a sequence of smooth functions defined on $\check{\mathcal{M}}$, satisfying $|\nabla_{\check{\mathcal{M}}}\psi| \leq C \cdot \kappa$, with*

$$\int_{\check{\mathcal{M}}} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} \leq C \cdot (\ln \kappa)^2. \tag{5.22}$$

Then, there exists a family of disjoint pseudo-balls $\hat{B}_j := \hat{B}(p_j, r_j)$ with $p_j \in \check{\mathcal{M}}$ for $j = 1, \dots, N_\kappa$, such that for κ sufficiently large

(1) $\{|\psi|^{-1}[0, 3/4)\} \subset \bigcup_{j=1}^{N_\kappa} \hat{B}_j$
(2) $N_\kappa \leq C \cdot (\ln \kappa)^2$
(3) $r_j \leq C \cdot (\ln \kappa)^{-6}$
(4) $\int_{\hat{B}_j} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right) \, d\mathcal{H}^2_{\check{\mathcal{M}}} \geq 2\pi \left|d^\kappa_j\right| (\ln \kappa - O(\ln \ln \kappa))$, *where* $d^{(\kappa)}_j = \deg(\psi_\kappa, \partial \hat{B}_j)$.

We will apply this proposition to global minimizers of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$. The hypotheses will be satisfied since, first of all, we can compare the energy of a minimizer to the energy of $\psi \equiv 1$ to get the energy bound (5.22). Then the estimate $|\nabla_{\check{\mathcal{M}}}\psi_\kappa| \leq C \cdot \kappa$ follows from elliptic regularity by working in local coordinates, rescaling these by $\frac{1}{\kappa}$ and applying standard Schauder theory, [13]. By the compactness of $\check{\mathcal{M}}$, one constant C can be obtained such that the estimate holds along the entire manifold.

First we need to adapt to our case the key estimates that this proposition is based upon. These estimates correspond to those in Theorem 2.1, [14], and its associated lemmas. In our situation we rephrase Lemma 2.4. of [14] in the following way:

Lemma 5.6. *Let $\psi \in H^1(\check{\mathcal{M}}; S^1)$. Then for any $\hat{B}(p, r) \subset \check{\mathcal{M}}$ and a.e. $r < \tilde{r}$ sufficiently small, one has*

$$\frac{2\pi}{r\,(1 + \mathcal{O}(r))} \deg(\psi, \partial \hat{B}(p, r))^2 \leq \int_{\partial \hat{B}(p,r)} |\nabla_{\check{\mathcal{M}}}\psi|^2 \, d\mathcal{H}^1. \tag{5.23}$$

Proof. Since ψ is S^1-valued, one can write $\psi = e^{i\Phi}$ and we have

$$\begin{aligned}\deg(\psi;\partial\,\hat{B}(p,r)) &= \frac{1}{2\pi}\int_{\partial\,\hat{B}(p,r)} \partial_\tau\Phi\,d\mathcal{H}^1\\ &\leq \frac{1}{2\pi}\int_{\partial\,\hat{B}(p,r)} |\partial_\tau\psi|\,d\mathcal{H}^1\\ &\leq \frac{1}{2\pi}\left(\int_{\partial\,\hat{B}(p,r)} |\partial_\tau\psi|^2\,d\mathcal{H}^1\right)^{\frac{1}{2}} \mathcal{H}^1\left(\partial\,\hat{B}(p,r)\right)^{1/2}\\ &\leq \frac{1}{2\pi}\left(\int_{\partial\,\hat{B}(p,r)} |\nabla_{\check{\mathcal{M}}}\psi|^2\,d\mathcal{H}^1\right)^{\frac{1}{2}} \left(2\pi r(1+\mathcal{O}(r))\right)^{\frac{1}{2}},\end{aligned}$$

where $\partial_\tau\cdot$ denotes tangential differentiation. The last line uses that the Jacobian of the exponential map is the identity at 0, as well as our analyticity assumptions, which in particular imply that the circumference of a pseudo-circle is of the order of $2\pi r(1+\mathcal{O}(r))$. The conclusion is immediate. □

We can now invoke the following lemmas, without major changes, from [14].

Lemma 5.7. *(cf. Lemma 2.5,* [14]*) Let $\psi \in H^1(\check{\mathcal{M}};\mathbb{C})$. There is a radius r_0 depending only on the surface such that for a.e. $r \in [\frac{1}{\kappa}, r_0)$ and for any $p \in \check{\mathcal{M}}$, one has*

$$\int_{\partial\,\hat{B}(p,r)} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\,d\mathcal{H}^1 \geq C\kappa(1-m)^N,$$

where $m := \inf_{\partial\,\hat{B}(p,r)}|\psi|$ and N is a positive constant independent of κ.

This result is then used to establish:

Lemma 5.8. *(cf. Theorem 2.1,* [14]*) If $\psi \in H^1(\check{\mathcal{M}};\mathbb{C})$, then for a.e. $r \in [\frac{1}{\kappa}, r_0)$ one has*

(5.24)

$$\int_{\partial\,\hat{B}(p,r)} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\,d\mathcal{H}^1 \geq \left(\frac{2\pi m^2}{r(1+\mathcal{O}(r))}\right)\deg(\psi,\partial\,\hat{B}(p,r))^2 + C\kappa(1-m)^N.$$

With these lemmas in hand, Proposition 5.5 will follow from the "growing balls" construction laid out in [21], Prop. 3.1. In our setting, we will only grow the balls up to a radius that remains significantly less that $\tilde{r}$ so that whenever two pseudo-balls are combined, they will sit inside a larger pseudo-ball. The only ingredient that is missing to perform this construction is a lower bound on the energy within a pseudo-annulus. This construction corresponds to the one carried out in Lemma 3.2 of [21]. In our case, it can be phrased through the following lemma.

Lemma 5.9. *Let $\hat{B}(p,r_1)$ and $\hat{B}(p,r_2)$ be two pseudo-balls centered at $p \in \check{\mathcal{M}}$, satisfying $\frac{1}{\kappa} < r_1 < r_2 < \tilde{r}$. Then there is a function $\Omega_\kappa : (0, \frac{2}{(\ln\kappa)^7}) \to \mathbb{R}$ satisfying the properties*

$$(i)\ \frac{\Omega_\kappa(s)}{s} \quad \textit{is decreasing}, \quad (ii)\ \sup \frac{\Omega_\kappa(s)}{s} \le C\kappa,$$

and (iii) there exists $\kappa_0 > 0$ such that if $\kappa > \kappa_0$ and $\frac{1}{\kappa} < t$ then

$$\left|\Omega_\kappa(t) - \pi \ln\left(t\kappa\right)\right| \le C.$$

Furthermore, whenever $|\psi| > \frac{3}{4}$ on $\hat{B}(p,r_2) \setminus \hat{B}(p,r_1)$, this function Ω_κ satisfies the following estimate:

$$\int_{\hat{B}(p,r_2)\setminus\hat{B}(p,r_1)} \left(|\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2\right) d\mathcal{H}^2_{\check{\mathcal{M}}} \ge |d| \left(\Omega_\kappa\left(\frac{r_2}{|d|}\right) - \Omega_\kappa\left(\frac{r_1}{|d|}\right)\right), \tag{5.25}$$

where we have written $d = \deg(\psi\,, \partial\,\hat{B}(p,r_2))$,

Proof. Defining $g : \hat{B}(p,r_2) \to \mathbb{R}$ via the formula

$$g(x) := \left\|\exp_p^{-1}(x)\right\|,$$

we note that since g simply measures distance along $\check{\mathcal{M}}$, one has $|\nabla_{\check{\mathcal{M}}} g(x)| = 1$. Also, since $\partial\,\hat{B}(p,r) = \exp_p(\partial\,B(p,r))$ and $\exp_p(\partial\,B(p,r)) = g^{-1}(r)$, we can apply the coarea formula and (5.24) to get

$$\begin{aligned}
&\int_{\hat{B}(p,r_2)\setminus\hat{B}(p,r_1)} \left(|\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\right) d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&= \int_{\hat{B}(p,r_2)\setminus\hat{B}(p,r_1)} \left(|\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\right) |\nabla_{\check{\mathcal{M}}} g| \, d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&\geq \int_{r_1}^{r_2}\int_{\partial\hat{B}(p,r)} \left(|\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2\right) d\mathcal{H}^1\, dr \\
&\geq \int_{r_1}^{r_2} \left(\frac{2\pi d^2 m^2}{r\big(1+\mathcal{O}(r)\big)} + C\kappa(1-m)^N\right) dr \\
&\geq \int_{r_1}^{r_2} \left(\frac{2\pi\,|d|\,m^{N_1}}{r\big(1+\mathcal{O}(r)\big)} + C\kappa(1-m)^{N_1}\right) dr \\
&\geq \int_{r_1}^{r_2} \left(\frac{2\pi\,|d|\,m^{N_1}}{r}\big(1-C_1 r\big) + C\kappa(1-m)^{N_1}\right) dr \\
&\geq \int_{r_1}^{r_2} \left(\frac{2\pi\,|d|\,m^{N_1}}{r}\left(1-\frac{C_1}{(\ln\kappa)^7}\right) + C\kappa(1-m)^{N_1}\right) dr,
\end{aligned}$$

where $N_1 := \max\{2, N\}$. The remainder of the argument follows as in the proof of Proposition 3.1 of [14] or in Lemma 3.2 of [21] by choosing $\Omega_\kappa(s)$ essentially as $\Lambda_{1/\kappa}(s)$ in these articles except that one adjusts the definition to accommodate the addition of the (harmless) term involving $\frac{C_1}{(\ln\kappa)^7}$.

□

Proof of Part II of Theorem 5.1

We will consider a sequence of global minimizers ψ_κ, of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$. We will drop the subscript throughout the rest of the proof for convenience and write simply ψ for ψ_κ. We begin by computing

$$\begin{aligned}
\Lambda((\mathbf{A}^e),\psi) \quad &= i\int_{\bigcup_{j\in I}\hat{B}_j} (\mathbf{A}^e)^\tau\cdot(\psi\nabla_{\check{\mathcal{M}}}\psi^* - \psi^*\nabla_{\check{\mathcal{M}}}\psi)\, d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&+ i\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j} (\mathbf{A}^e)^\tau\cdot(\psi\nabla_{\check{\mathcal{M}}}\psi^* - \psi^*\nabla_{\check{\mathcal{M}}}\psi)\, d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&= I + II. \qquad (5.26)
\end{aligned}$$

Note that from (5.1), Proposition 5.5, (3) and comparison with the energy of the function 1, an application of Hölder's inequality yields

$$
\begin{aligned}
|I| &\leq C\,\|\mathbf{A}^e\|_{L^\infty}\,\|\nabla_{\check{\mathcal{M}}}\psi\|_{L^2(\check{\mathcal{M}})}\cdot\frac{N_\kappa}{(\ln\kappa)^6}\\
&\leq C\big(\ln\kappa\big)\,\big(\ln\kappa\big)\,\frac{\big(\ln\kappa\big)^2}{(\ln\kappa)^6}\leq\frac{C}{(\ln\kappa)^2}.
\end{aligned}
\tag{5.27}
$$

To estimate II, we substitute in $\alpha := \frac{\psi}{|\psi|}$ and find that

$$
\begin{aligned}
II &= i\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}(\mathbf{A}^e)^\tau\cdot(\alpha\nabla_{\check{\mathcal{M}}}\alpha^*-\alpha^*\nabla_{\check{\mathcal{M}}}\alpha)\;d\mathcal{H}^2_{\check{\mathcal{M}}}\\
&\quad+i\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}(|\psi|^2-1)\,(\mathbf{A}^e)^\tau\cdot(\alpha\nabla_{\check{\mathcal{M}}}\alpha^*-\alpha^*\nabla_{\check{\mathcal{M}}}\alpha)\;d\mathcal{H}^2_{\check{\mathcal{M}}}\\
&= III+IV.
\end{aligned}
\tag{5.28}
$$

Again comparing the energy of ψ to the energy of the function 1, and using the fact that $|\psi|\geq 3/4$ on $\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j$ (cf. Prop. 5.5, (1)), we can estimate IV as follows:

$$
\begin{aligned}
|IV| &\leq 2\,\|\mathbf{A}^e\|_{L^\infty}\left(\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}(|\psi|^2-1)^2\,d\mathcal{H}^2_{\check{\mathcal{M}}}\right)^{1/2}\left(\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}|\nabla_{\check{\mathcal{M}}}\alpha|^2\;d\mathcal{H}^2_{\check{\mathcal{M}}}\right)^{1/2}\\
&\leq 2\,\|\mathbf{A}^e\|_{L^\infty}\left(\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}(|\psi|^2-1)^2\,d\mathcal{H}^2_{\check{\mathcal{M}}}\right)^{1/2}\left((4/3)^2\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}|\psi|^2\,|\nabla_{\check{\mathcal{M}}}\alpha|^2\;d\mathcal{H}^2_{\check{\mathcal{M}}}\right)^{1/2}\\
&\leq C\big(\ln\kappa\big)\left(\frac{\ln\kappa}{\kappa}\right)\left(\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}|\nabla_{\check{\mathcal{M}}}\psi|^2\;d\mathcal{H}^2_{\check{\mathcal{M}}}\right)^{1/2}\leq C\left(\frac{(\ln\kappa)^3}{\kappa}\right).
\end{aligned}
\tag{5.29}
$$

Combining (5.26)–(5.29), we find that

$$
\Lambda((\mathbf{A}^e)^\tau,\psi)=i\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j}(\mathbf{A}^e)^\tau\cdot(\alpha\nabla_{\check{\mathcal{M}}}\alpha^*-\alpha^*\nabla_{\check{\mathcal{M}}}\alpha)\;d\mathcal{H}^2_{\check{\mathcal{M}}}+\mathcal{O}\bigg(\frac{1}{(\ln\kappa)^2}\bigg).
\tag{5.30}
$$

Now we let $F:[0,\pi]\to\mathbb{R}$ denote any primitive of $u\gamma$ and then define $F_{\check{\mathcal{M}}}:\check{\mathcal{M}}\to\mathbb{R}$ via $F_{\check{\mathcal{M}}}(x):=F(\phi(x))$. This, of course, only determines $F_{\check{\mathcal{M}}}$ up to a constant, but as we shall see in the calculation to follow, it is only the difference max $F_{\check{\mathcal{M}}}-$ min $F_{\check{\mathcal{M}}}$ that will matter and one readily computes from (4.8) that

$$\max F_{\check{\mathcal{M}}} - \min F_{\check{\mathcal{M}}} = \frac{\mathcal{H}^2(\check{\mathcal{M}})}{2\pi}. \tag{5.31}$$

Then by (4.8), (4.9) and exterior differentiation we obtain

$$\begin{aligned} &i\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j} (\mathbf{A}^e)^\tau\cdot(\alpha\nabla_{\check{\mathcal{M}}}\alpha^* - \alpha^*\nabla_{\check{\mathcal{M}}}\alpha)\ d\mathcal{H}^2_{\check{\mathcal{M}}} \\ &= i\frac{h(\kappa)}{2}\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j} (\alpha\, dF_{\check{\mathcal{M}}}\wedge d\alpha^* - \alpha^*\, dF_{\check{\mathcal{M}}}\wedge d\alpha) \\ &= i\frac{h(\kappa)}{2}\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j} d\big(F_{\check{\mathcal{M}}}(\alpha d\alpha^* - \alpha^* d\alpha)\big) + i\frac{h(\kappa)}{2}\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j} F_{\check{\mathcal{M}}}(d\alpha^*\wedge d\alpha - d\alpha\wedge d\alpha^*). \end{aligned} \tag{5.32}$$

The last integral is zero since $|\alpha| = 1$, and so integration by parts on the penultimate integral yields

$$\begin{aligned} &i\int_{\check{\mathcal{M}}\setminus\bigcup_{j\in I}\hat{B}_j} (\mathbf{A}^e)^\tau\cdot(\alpha\nabla_{\check{\mathcal{M}}}\alpha^* - \alpha^*\nabla_{\check{\mathcal{M}}}\alpha)\ d\mathcal{H}^2_{\check{\mathcal{M}}} = i\frac{h(\kappa)}{2}\sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j} F_{\check{\mathcal{M}}}\ (\alpha d\alpha^* - \alpha^* d\alpha) \\ &= \frac{h(\kappa)}{2}\sum_{j=1}^{N_\kappa} F_{\check{\mathcal{M}}}(p_j)\int_{\partial\hat{B}_j} i\,(\alpha d\alpha^* - \alpha^* d\alpha) + \frac{h(\kappa)}{2}\sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j}\big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) i\,(\alpha d\alpha^* - \alpha^* d\alpha) \\ &= 2\pi h(\kappa)\sum_{j=1}^{N_\kappa} F_{\check{\mathcal{M}}}(p_j) d_j^{(\kappa)} + \frac{h(\kappa)}{2}\sum_{j=1}^{N_\kappa}\int_{\partial\hat{B}_j}\big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) i\,(\alpha d\alpha^* - \alpha^* d\alpha). \end{aligned} \tag{5.33}$$

To control the last sum in (5.33) we define

$$\hat{\psi} := \begin{cases} \psi & \text{if } |\psi| \le 3/4, \\ \frac{3}{4}\frac{\psi}{|\psi|} & \text{if } |\psi| > 3/4 \end{cases}$$

and $\hat{\alpha} := \frac{\hat{\psi}}{|\hat{\psi}|}$. Then we compute

$$
\begin{aligned}
\sum_{j=1}^{N_\kappa} \int_{\partial \hat{B}_j} \big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) i\,(\alpha d\alpha^* - \alpha^* d\alpha) &= \sum_{j=1}^{N_\kappa} \int_{\partial \hat{B}_j} \big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) i\,(\hat{\alpha} d\hat{\alpha}^* - \hat{\alpha}^* d\hat{\alpha}) \\
&= \frac{16}{9} \sum_{j=1}^{N_\kappa} \int_{\partial \hat{B}_j} \big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) i\,(\hat{\psi} d\hat{\psi}^* - \hat{\psi}^* d\hat{\psi}) \\
&= \frac{16}{9} \sum_{j=1}^{N_\kappa} \int_{\hat{B}_j} d\Big(\big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) i\,(\hat{\psi} d\hat{\psi}^* - \hat{\psi}^* d\hat{\psi})\Big) \\
&= \frac{16}{9} \sum_{j=1}^{N_\kappa} \int_{\hat{B}_j} d\, F_{\check{\mathcal{M}}} \wedge i\,\big(\hat{\psi} d\hat{\psi}^* - \hat{\psi}^* d\hat{\psi}\big) \\
&+ \frac{32}{9} \sum_{j=1}^{N_\kappa} \int_{\hat{B}_j} \big(F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)\big) d\,\hat{\psi} \wedge d\,\hat{\psi}^* = I_1 + I_2. \qquad (5.34)
\end{aligned}
$$

But since the gradient of $F_{\check{\mathcal{M}}}$ is bounded on $\check{\mathcal{M}}$ and the norm of the gradient of $\hat{\psi}$ is bounded by the norm of the gradient of ψ, we can invoke Proposition 5.5 to find that

$$
\begin{aligned}
h(\kappa)\,|I_1| &\le C h(\kappa)\, \|\nabla_{\check{\mathcal{M}}} F_{\check{\mathcal{M}}}\|_{L^\infty(\check{\mathcal{M}})} \sum_{j=1}^{N_\kappa} \|\nabla_{\check{\mathcal{M}}} \psi\|_{L^2(\hat{B}_j)}\, \|1\|_{L^2(\hat{B}_j)} \\
&\le C\,\big(\ln\kappa\big)\, \|\nabla_{\check{\mathcal{M}}} F_{\check{\mathcal{M}}}\|_{L^\infty(\check{\mathcal{M}})} \|\nabla_{\check{\mathcal{M}}} \psi\|_{L^2(\check{\mathcal{M}})}\, N_\kappa \frac{1}{(\ln\kappa)^6} \le C \frac{\big(\ln\kappa\big)^4}{(\ln\kappa)^6}. \qquad (5.35)
\end{aligned}
$$

Similarly, since $|F_{\check{\mathcal{M}}} - F_{\check{\mathcal{M}}}(p_j)| \le \frac{C}{(\ln\kappa)^6}$ inside $\hat{B}_j$, we see that

$$
h(\kappa)\,|I_2| \le C\,\big(\ln\kappa\big)\, \|\nabla_{\check{\mathcal{M}}} \psi\|^2_{L^2(\check{\mathcal{M}})} \frac{N_\kappa}{(\ln\kappa)^6} \le C\, \frac{\big(\ln\kappa\big)^5}{(\ln\kappa)^6}. \qquad (5.36)
$$

Hence, the last term in (5.33) is indeed $o(1)$.

Combining (5.30), (5.32), (5.33) (5.34), (5.35) and (5.36) we conclude that

$$
\Lambda(\mathbf{A}^e, \psi) = 2\pi h(\kappa) \sum_{j=1}^{N_\kappa} F_{\check{\mathcal{M}}}(p_j) d_j^{(\kappa)} + o(1). \qquad (5.37)
$$

Next, we note that since $\check{\mathcal{M}}$ is closed, the total degree is zero, that is:

$$
4\pi \Sigma_{j\in I} d_j^{(\kappa)} = i \int_{\check{\mathcal{M}} \setminus \bigcup_{j\in I} \hat{B}_j} d(\alpha d\alpha^* - \alpha^* d\alpha) = 0. \qquad (5.38)
$$

Denoting by N_κ^+ the number of pseudo-balls out of the total of N_κ that carry a positive degree, and assuming, without any loss of generality, that the pseudo-balls are ordered so that the ones with positive degree are listed first, we can express (5.38) as

$$\sum_{j=1}^{N_\kappa^+} d_j^{(\kappa)} + \sum_{j=N_\kappa^++1}^{N_\kappa} d_j^{(\kappa)} = 0 \tag{5.39}$$

or equivalently,

$$\sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| = 2 \sum_{j=1}^{N_\kappa^+} d_j^{(\kappa)}. \tag{5.40}$$

Now we use Proposition 5.5, (5.19) and (5.37) to estimate the energy of a minimizer ψ from below as

$$\begin{aligned}
\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi) &= \int_{\check{\mathcal{M}}} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} + \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau|^2 \, |\psi|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} - \Lambda(\mathbf{A}^e,\psi) \\
&\geq \sum_{j=1}^{N_\kappa} \int_{\hat{B}(p_j,r_j)} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} \\
&+ \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} - 2\pi h(\kappa) \sum_{j=1}^{N_\kappa} F_{\check{\mathcal{M}}}(p_j) d_j^{(\kappa)} - o(1) \\
&\geq 2\pi \sum_{j=1}^{N_\kappa} \left| d_j^{(\kappa)} \right| (\ln\kappa - \mathcal{O}(\ln\ln\kappa)) + \left(\frac{h(\kappa)}{2}\right)^2 \|u\|^2_{L^2(\check{\mathcal{M}})} \\
&-2\pi h(\kappa) \sum_{j=1}^{N_\kappa} F_{\check{\mathcal{M}}}(p_j) d_j^{(\kappa)} \; - o(1).
\end{aligned} \tag{5.41}$$

Appealing once again to the comparison $\mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi) \leq \mathcal{G}_{\check{\mathcal{M}},\kappa}(1)$, cf. (5.9), we can then invoke (5.31), (5.39) and (5.40) to conclude that

$$\begin{aligned}
\sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| (\ln\kappa - \mathcal{O}(\ln\ln\kappa)) &\leq h(\kappa) \sum_{j=1}^{N_\kappa} F_{\check{\mathcal{M}}}(p_j) d_j^{(\kappa)} + o(1) \\
&\leq h(\kappa)\big(\max_{\check{\mathcal{M}}} F_{\check{\mathcal{M}}}\big) \sum_{j=1}^{N_\kappa^+} d_j^{(\kappa)} + h(\kappa)\big(\min_{\check{\mathcal{M}}} F_{\check{\mathcal{M}}}\big) \sum_{j=N_\kappa^+ +1}^{N_\kappa} d_j^{(\kappa)} + o(1) \\
&= \left(h(\kappa)\sum_{j=1}^{N_\kappa^+} d_j^{(\kappa)}\right)\left(\max_{\check{\mathcal{M}}} F_{\check{\mathcal{M}}} - \min_{\check{\mathcal{M}}} F_{\check{\mathcal{M}}}\right) + o(1) \\
&= \frac{\mathcal{H}^2(\check{\mathcal{M}})}{4\pi} h(\kappa) \sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| + o(1).
\end{aligned}$$

But in view of (5.1) and the assumption $C_0 < \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$, this cannot hold for κ sufficiently large unless

$$\sum_{j=1}^{N_\kappa} \left|d_j^{(\kappa)}\right| = 0, \tag{5.42}$$

that is, unless any zeros of the minimizer ψ have zero degree.

Pursuing this possibility, however, we note that (5.37) would then imply that $\Lambda(\mathbf{A}^e,\psi) = o(1)$ and so in view of (5.19) we would find

$$\begin{aligned}
\int_{\check{\mathcal{M}}} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2 d\mathcal{H}^2_{\check{\mathcal{M}}} &= \mathcal{G}_{\check{\mathcal{M}},\kappa}(\psi) - \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau|^2 |\psi|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} + o(1) \\
&\leq \mathcal{G}_{\check{\mathcal{M}},\kappa}(1) - \int_{\check{\mathcal{M}}} |(\mathbf{A}^e)^\tau|^2 |\psi|^2 \, d\mathcal{H}^2_{\check{\mathcal{M}}} + o(1) = o(1).
\end{aligned} \tag{5.43}$$

But if there exists even one zero of ψ of zero degree, say at $x = p \in \check{\mathcal{M}}$, then the estimate $|\nabla_{\check{\mathcal{M}}}\psi| \leq C\cdot\kappa$ implies that $|\psi| \leq 1/2$ on a pseudo-ball $\hat{B}(p,r)$ for a radius $r \geq \frac{C_1}{\kappa}$ for some C_1 independent of κ. Hence, we can rule out this possibility since we would then have

$$\int_{\check{\mathcal{M}}} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2 d\mathcal{H}^2_{\check{\mathcal{M}}} \geq \int_{\hat{B}(p,r)} |\nabla_{\check{\mathcal{M}}}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2-1\right)^2 d\mathcal{H}^2_{\check{\mathcal{M}}} \geq C_2,$$

for some positive constant C_2 independent of κ, contradicting (5.43). □

5.3. **Existence and non-existence of vortices for $G_{\varepsilon,\kappa}$.** We conclude with results on the existence and non-existence of vortices for minimizers of the 3d Ginzburg-Landau functional $G_{\varepsilon,\kappa}$ given by (2.2). These will follow from the asymptotic value of H_{c1} associated with the energy $\mathcal{G}_{\check{\mathcal{M}},\kappa}$ for large κ established in Theorem 5.1, combined with the Γ-convergence shown in Theorem 3.1 and the compactness demonstrated in Proposition 3.5.

Proposition 5.10. *Assume $C_0 > \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$ in condition* (5.1)*. Fix any value $\kappa \geq \kappa_0$ where κ_0 is the value arising in Theorem 5.1, Part I, and for any $\varepsilon > 0$, let Ψ_ε denote a minimizer of $G_{\varepsilon,\kappa}$. Then there exists a value $\varepsilon_0 = \varepsilon_0(\kappa)$ such that for all positive $\varepsilon < \varepsilon_0$ there exists a circle $C_{\phi_\varepsilon} \subset \check{\mathcal{M}}$ as defined in* (5.2) *satisfying the condition*

$$\Psi_\varepsilon \neq 0 \text{ everywhere on } C_{\phi_\varepsilon,t}, \quad \deg\left(\Psi_\varepsilon, C_{\phi_\varepsilon,t}\right) \neq 0 \quad \text{for all} \quad t \in (0,1) \tag{5.44}$$

where

$$C_{\phi_\varepsilon,t} := \{x + \varepsilon t \nu(x) : \; x \in C_{\phi_\varepsilon}\} \subset \check{\mathcal{M}}_{\varepsilon,t},$$

cf. (2.7)*. Hence, in particular, Ψ_ε vanishes at least twice on each manifold $\check{\mathcal{M}}_{\varepsilon,t}$, for $0 < t < 1$.*

Furthermore, if $\{\varepsilon_j\} \to 0$ is a subsequence such that $\psi_{\varepsilon_j,\kappa} \to \psi_\kappa$ in $C^{0,\alpha}\big(\check{\mathcal{M}} \times (0,1)\big)$ as in Proposition 3.5, for some minimizer ψ_κ of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$, then for each of the two vortices of ψ_κ, say $p_1, \; p_2 \in \check{\mathcal{M}}$ guaranteed by Theorem 5.1, Part I, and for all $t \in (0,1)$, there exist sequences $\{p_1^j(t)\}$ and $\{p_2^j(t)\}$ of zeros of $\Psi_{\varepsilon_j\kappa}$ lying on $\check{\mathcal{M}}_{\varepsilon_j,t}$ such that $p_1^j(t) \to p_1$ and $p_2^j(t) \to p_2$ as $j \to \infty$.

Proof. Suppose, by way of contradiction, that the assertion (5.44) does not hold along a sequence $\{\varepsilon_j\} \to 0$. After perhaps passing to a further subsequence (still denoted by ε_j), we may apply Proposition 3.5 to establish that $\psi_{\varepsilon_j,\kappa} \to \psi_\kappa$ in $C^{0,\alpha}$, where ψ_κ is a minimizer of $\mathcal{G}_{\check{\mathcal{M}},\kappa}$. Associated with this minimizer there is a value $\phi \in (0,\pi)$ guaranteed by Theorem 5.1, Part I such that (5.3) holds along $C_\phi \subset \check{\mathcal{M}}$. Since ψ_κ is independent of t, (5.3) must also hold for ψ_κ where C_ϕ replaced by $C_{\phi,t} \subset \check{\mathcal{M}}_{\varepsilon,t}$ and t is any value in $(0,1)$. But then (5.44) must be valid for $\Psi_{\varepsilon_j,\kappa}$ in light of the uniform convergence, and a contradiction is reached. The second assertion of Corollary 5.10 also follows immediately from the uniform convergence of $\psi_{\varepsilon_j,\kappa}$ to ψ_κ, the t-independence of ψ_κ and the fact that the zeros of ψ_κ guaranteed by Theorem 5.1, Part I are isolated and have nonzero degree. □

Proposition 5.11. *Assume* $C_0 < \frac{4\pi}{\mathcal{H}^2(\check{\mathcal{M}})}$ *in condition* (5.1)*. Fix any value* $\kappa \geq \kappa_0$ *where* κ_0 *is the value arising in Theorem 5.1, Part II and for any* $\varepsilon > 0$*, let* $\Psi_{\varepsilon,\kappa}$ *denote a minimizer of* $G_{\varepsilon,\kappa}$*. Then there exists a value* $\varepsilon_0 = \varepsilon_0(\kappa)$ *such that for all positive* $\varepsilon < \varepsilon_0$*,* $\Psi_{\varepsilon,\kappa}$ *does not vanish in* Ω_ε*.*

Proof. This result is an immediate consequence of the uniform convergence of minimizers of $G_{\varepsilon,\kappa}$ guaranteed by Proposition 3.5, coupled with the non-vanishing property of minimizers of the Γ-limit provided by Theorem 5.1, Part II. □

Propositions 5.10 and 5.11 together in particular imply Theorem 5.3.

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