ANY LATTICE CAN BE REGULARLY EMBEDDED INTO THE MACNEILLE COMPLETION OF A DISTRIBUTIVE LATTICE

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The completion by cuts of a totally ordered set was first introduced by Dedekind in his famous construction of the real numbers from the rationals. MacNeille [4] extended the method of completion by cuts to arbitrary partially ordered sets. For a partially ordered set $P$ and $A \subseteq P$, defining

$$U(A) = \{x \in P : x \geq y \text{ for all } y \in A\}$$
$$L(A) = \{x \in P : x \leq y \text{ for all } y \in A\}$$

a cut, or normal ideal, of $P$ is a subset $A$ of $P$ for which $A = LU(A)$. The set of all normal ideals of $P$ partially ordered by set inclusion forms a complete lattice $\bar{P}$ where the supremum $\bigvee$ and infinum $\bigwedge$ of a subset $S$ of $\bar{P}$ are given by

$$\bigvee S = LU \left( \bigcup S \right) \quad \text{and} \quad \bigwedge S = \bigcap S$$

The partially ordered set $P$ can be embedded into its MacNeille completion $\bar{P}$ and this embedding is both supremum and infimum dense. That is to say that every element of $\bar{P}$ is the supremum of elements in the image of $P$ and the infimum of elements in the image of $P$. It has been shown (see[1,5]) that any complete lattice into which $P$ can be supremum and infimum densely embedded is isomorphic to the MacNeille completion of $P$.

It is well known that the MacNeille completion is not particularly well behaved with respect to preserving lattice identities. Funayama [3]

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has shown that the MacNeille completion of a distributive lattice need not even be modular. His proof follows by showing that the non-modular five element lattice $N_5$ can be embedded into the MacNeille completion of a distributive lattice. It is the purpose of this paper to give the following result. Recall that a regular embedding is an embedding that preserves all existing joins and meets.

**Theorem.** Any lattice can be regularly embedded into the MacNeille completion of a distributive lattice.

We begin by constructing the various objects required in the proof. For a lattice $L$, define the following for each $y \in L$ and each integer $m$ (the set of integers will be denoted by $\mathbb{Z}$).

- $P$ is the set of all non-empty finite subsets of $L$.
- $X = L \times P \times \mathbb{Z}$.
- $X_y$ is the slab of $X$ for which the first components are dominated by $y$, i.e. $X_y = \text{pr}_L^{-1}[\leftarrow, y]$.
- $X^m_y = \{(x, A, n) \in X : x \leq y, n \leq m\} \cup \{(x, A, n) : x \text{ is a zero of } L\}$.
- $U^m_y = \{(x, A, n) \in X : y \notin A\} \cup \{(x, A, n) \in X : x \leq \bigvee A\} \cup \{(x, A, n) \in X : n \leq m\}$.
- $D$ is the sublattice of the power set lattice of $X$ generated by $\{X^m_y, U^m_y : y \in L, m \in \mathbb{Z}\}$. In particular, $D$ is distributive.
- $N_y$ is the collection of all elements of $D$ which are contained in $X_y$.

In particular, $N_y$ is a non-empty ideal of $D$.

Before diving into the details of the proof, let me give a brief outline of the plan. As might be guessed by the notation, the sets $N_y$ are intended to be normal ideals of $D$. For each $y \in L$, the collection of sets $\{U^m_y\}_{m \in \mathbb{Z}}$ is intended to serve as a set of upper bounds of $N_y$, refined enough to ensure that $N_y$ is a normal ideal of $D$. For a finite non-empty subset $A$ of $L$, to have an embedding of $L$ into $D$ we want the supremum of $\{N_a : a \in A\}$ in $D$ to be $N_{VA}$. The essential point is that $\bigcup_{a \in A} U^m_y$ should be an upper bound of $N_{VA}$. Intuitively this says that the sets $U^m_y$ are reasonably full. It
is the dual role played by the sets $U_y^m$ that necessitates their complicated definition.

To simplify notation in the following Lemma, let $G$ denote the set of generators of the lattice $D$, i.e. $G = \{X_y^m, U_y^m : y \in L, m \in \mathcal{Z}\}$. Also, let $G_\cap$ and $G_\cup$ be the closure of $G$ under finite non-empty intersections and finite non-empty unions respectively. As $D$ is a distributive lattice generated by $G$, any element of $D$ can be expressed as a finite non-empty union of elements of $G_\cap$ or dually, as a finite non-empty intersection of members of $G_\cup$.

A set $G$ in $G_\cap$ has a representation as

$$G = \bigcap_{i=1}^{p} X_{a_i}^{m_i} \cap \bigcap_{j=1}^{q} U_{b_j}^{n_j}$$

where $p, q$ are positive integers, not both 0, and $a_i, b_j \in L, m_i, n_j \in \mathcal{Z}$ for each $1 \leq i \leq p, 1 \leq j \leq q$. It is not difficult to see that if $a_i = a_k$ for some $1 \leq i, k \leq p$ then one of the terms $X_{a_i}^{m_i}, X_{a_k}^{m_k}$ is redundant. Following this reasoning, we can represent $G$ by

$$G = \bigcap_{a \in A} X_a^{m_a} \cap \bigcap_{b \in B} U_b^{n_b}$$

where $A, B$ are finite subsets of $L$, not both empty, and $m_a, n_b$ are integers for each $a \in A, b \in B$. Of course, similar statements hold for $G_\cup$.

We will have need to use such representations frequently, and as no confusion is possible as to the nature of the entities $A, B, m_a, n_b$, references to their nationalities will be omitted.

**Lemma 1.**

i) For $a, b \in L$ and $n, m$ integers, $X_a^m \cap X_b^n = X_{a \wedge b}^{\min\{n, m\}}$.

ii) If $y$ is a zero of $L$ then $X_y \subseteq X_a^m$ and $X_y \subseteq U_a^m$ for each $a \in L, m \in \mathcal{Z}$.

iii) If $A, B$ are finite subsets of $L$, not both empty, then

$$\bigcup_{a \in A} X_a^{m_a} \cup \bigcup_{b \in B} U_b^{n_b} \supseteq X_y \text{ if and only if } \bigvee B \text{ exists and } \bigvee B \geq y.$$ 

iv) For $G \in G_\cup$ and $S \subseteq L$, if $\forall S$ exists and $G \supseteq \bigcup_{s \in S} X_s$ then $G \supseteq X_{\forall S}$.

v) For $y \in L$ and $G \in G_\cap$, if $G \subseteq U_y^m$ for each integer $m$, then $G \subseteq X_y$. 
Proof.

i) This is a straight forward calculation.

ii) For \((x, A, n) \in X_y\), if \(y\) is a zero of \(L\) then \(x = y\), so \(x\) is a zero of \(L\) and \(x \leq \vee A\) giving \((x, A, n) \in X^m_a\) and \((x, A, n) \in U^m_a\).

iii) First we check that special case that \(y\) is a zero of \(L\). By part ii) we have \(X_y \subseteq \bigcup_{a \in A} X^m_a \cup \bigcup_{b \in B} U^{m_b}_b\) since not both of \(A, B\) are empty. As \(y\) is a zero of \(L\), even if \(B\) is empty \(B\) has a supremum in \(L\) and \(\forall B \geq y\).

Assume that \(y\) is not a zero of \(L\) and that
\[
\bigcup_{a \in A} X^m_a \cup \bigcup_{b \in B} U^{m_b}_b \supseteq X_y.
\]
This implies that \(B\) must be non-empty. Setting \(t = \max\{m_a, n_b : a \in A, b \in B\} + 1\), we have \((y, B, t) \in X_y\), so for some \(b \in B\) we have \((y, B, t) \in U^{m_b}_b\). Then either \(b \notin B\), or \(t \leq n_b\) or \(\forall B \geq y\). The first two conditions are obviously false, giving \(\forall B \geq y\).

Assume that \(y\) is not a zero of \(L\), that \(\forall B\) exists and that \(\forall B \geq y\). This implies that \(B\) is non-empty. Take \((x, C, p) \in X_y\) and consider two cases; that \(B\) is contained in \(C\) and that \(B\) is not contained in \(C\). In the first case \(\forall C \geq \forall B \geq y \geq x\), giving that \((x, C, p) \in U^{m_b}_b\) for each \(b \in B\). In the second case, there is some element \(b \in B\) with \(b \notin C\), giving \((x, C, p) \in U^{m_b}_b\). So, \(X_y \subseteq \bigcup_{b \in B} U^{m_b}_b\).

iv) As \(G \in G_U\), there is a representation
\[
G = \bigcup_{a \in A} X^m_a \cup \bigcup_{b \in B} U^{m_b}_b
\]
where not both of \(A, B\) are empty. If \(G \supseteq \bigcup_{s \in S} X_s\) then by part iii) we have that \(\forall B\) exists and \(\forall B \geq s\) for each \(s \in S\). If \(\forall S\) also exists, then \(\forall B \geq \forall S\) and so by part iii) \(G \supseteq X_{\forall S}\).

v) As \(G \in G_n\), there is a representation
\[
G = \bigcap_{a \in A} X^m_a \cap \bigcap_{b \in B} U^{m_b}_b
\]
where not both of \(A, B\) are empty. By part i) we may assume that \(A\) has at most one element.

If \(y\) is a unit of \(L\) then \(X_y = X\) so clearly \(G \subseteq X_y\). Assume then that \(y\) is not a unit of \(L\) and that \(z \notin y\).
If $B$ is non-empty, setting $t = \min \{ n_b : b \in B \}$ we have $(z, \{ y \}, t) \in \cap_{b \in B} U_{n_b}^t$. But $z \notin \bigvee \{ y \}$, so $(z, \{ y \}, t) \notin U_{\bigvee \{ y \}}^{t-1}$.

If $G \subseteq U_y^m$ for each integer $m$, from the above remarks we may conclude that $A$ is non-empty, and consists of a single element, say $a$. Setting $p = \min \{ m_a, n_b : b \in B \}$, we have that $(a, \{ y \}, p) \in G$ so $(a, \{ y \}, p) \in U_y^m$ for each integer $m$. However, this can only occur if $a \leq \bigvee \{ y \} = y$ giving that $G \subseteq X_a^{m_a} \subseteq X_y$.

**Lemma 2.** For each $y \in L$, $N_y$ is a normal ideal of $D$, and if $y \neq z$ then $N_y \neq N_z$.

**Proof.** We must show that $N_y = LU(N_y)$. From general principles it follows that $N_y \subseteq LU(N_y)$. Note that by applying part iii) of Lemma 1 for the special case of $A$ being empty and $B = \{ y \}$, we have that $U_y^m \supseteq X_y$ for each integer $m$. So $U_y^m$ is an upper bound of $N_y$ for each integer $m$. Suppose $G \in LU(N_y)$ and that $G = G_1 \cup \ldots \cup G_n$, with $n \geq 1$, is a representation of $G$ as a union of members of $G_n$. Then for each $1 \leq i \leq n$ we have $G_i \in LU(N_y)$ and in particular $G_i \subseteq U_y^m$ for each integer $m$. Then by part v) of Lemma 1, for each $1 \leq i \leq n$ we have $G_i \subseteq N_y$ for each integer $m$. Then by part v) of Lemma 1, for each $1 \leq i \leq n$ we have $G_i \subseteq X_y$ so $G_i \in N_y$. Then as $N_y$ is an ideal of $D$, $G \in N_y$.

To see the further remark, note that if $y \notin z$ then $X_y^1 \in N_y$ but $X_y^1 \notin N_z$.

**Lemma 3.** If $S \subseteq L$ and $\wedge S$ exists then $N_{\wedge S} = \cap_{s \in S} N_s$.

**Proof.**

Note that $\bigcap_{s \in S} X_s = \{ (x, A, m) \in X : x \leq s \text{ for all } s \in S \}$

$$= \{ (x, A, m) \in X : x \leq \wedge S \}$$

$$= X_{\wedge S}.$$  

So $\bigcap_{s \in S} N_s = \{ G \in D : G \subseteq X_s \text{ for each } s \in S \}$

$$= \{ G \in D : G \subseteq \bigcap_{s \in S} X_s \}$$

$$= \{ G \in D : G \subseteq X_{\wedge S} \}$$

$$= N_{\wedge S}$$
Lemma 4. If $S \subseteq L$ and $\bigvee S$ exists then $N_{\bigvee S} = LU(\bigcup_{s \in S} N_s)$.

Proof. It will be sufficient to show that $U(N_{\bigvee S}) = U(\bigcup_{s \in S} N_s)$ since this statement implies that $LU(N_{\bigvee S}) = LU(\bigcup_{s \in S} N_s)$ and Lemma 2 has supplied the fact that $N_{\bigvee S}$ is a normal ideal of $D$ so $LU(N_{\bigvee S}) = N_{\bigvee S}$.

As $N_{\bigvee S}$ contains $\bigcup_{s \in S} N_s$, it follows that $U(N_{\bigvee S}) \subseteq U(\bigcup_{s \in S} N_s)$. Suppose $G$ is an upper bound of $\bigcup_{s \in S} N_s$ and that $G = G_1 \cap \cdots \cap G_n$, where $n \geq 1$, is a representation of $G$ as a finite intersection of members of $\mathcal{G}_U$. Then for each $1 \leq i \leq n$ we have that $G_i$ is an upper bound of $\bigcup_{s \in S} N_s$, so $G_i \supseteq \bigcup_{s \in S} X_s$. Then by part iv) of Lemma 1, for each $1 \leq i \leq n$ we have that $G_i \supseteq X_{\bigvee S}$. So $G \supseteq X_{\bigvee S}$ and $G$ is an upper bound of $N_{\bigvee S}$.

Lemmas 2, 3, and 4 show that the map which sends an element $y$ of $L$ to the subset $N_y$ of $D$ is a regular embedding of $L$ into the MacNeille completion of the distributive lattice $D$.

REFERENCES


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