Order **10**: 283–294, 1993.

Completions of Orthomodular Lattices II

JOHN HARDING*
Department of Mathematics, Vanderbilt University, Nashville, Tennessee, 37240, U.S.A.

Communicated by B. Jónsson

(Received: 21 July 1992; accepted: 5 May 1993)

Abstract. If $\mathcal{K}$ is a variety of orthomodular lattices generated by a set of orthomodular lattices having a finite uniform upper bound om the length of their chains, then the MacNeille completion of every algebra in $\mathcal{K}$ again belongs to $\mathcal{K}$.

Mathematics Subject Classifications (1991). 06A10, 20B14.

Key words: Orthomodular lattices, MacNeille completion, Boolean algebra.

It is often of interest to know when the fundamental operations of an ordered structure A can be extended to operations of the MacNeille completion of A, subject to certain constraints. As an important example, the MacNeille completion of a Boolean algebra B carries the structure of a Boolean algebra, which is the injective hull of B [10]. Other examples naturally occur; the MacNeille completion of a Heyting algebra H can be given a Heyting algebra structure extending that of H ([2], p. 238), the MacNeille completion of an ortholattice L can be given an ortholattice structure extending that of L [18], and of course, the MacNeille completion of a lattice L can be given a lattice structure extending that of L [19]. For a variety V of algebras whose members carry a natural partial ordering, we say that V is closed under MacNeille completions if the operations of an algebra A in V can be extended to the MacNeille completion of A so that the resultant belongs in V. So that the examples given above do not mislead anyone, it is shown in [13] that the only varieties of lattices which are closed under MacNeille completions are the trivial variety and the variety of all lattices.

In [4], it was shown that any variety of orthomodular lattices which is generated by a single finite orthomodular lattice is closed under MacNeille completions, in contrast to the fact that the variety of all orthomodular lattices is not closed under MacNeille completions [1, 11]. In this paper we extend the results of [4] to show that any variety of orthomodular lattices which is generated by a set of orthomodular lattices having a finite uniform upper bound on the lengths of their chains is closed under MacNeille completions. A natural example of such a variety is one generated by a set of n-dimensional orthocomplemented projective geometries. Our approach has the added advantage of constructing the MacNeille completion

* The author gratefully acknowledges the support of NSERC.

of an algebra A in such a variety as a subalgebra of a reduced product of quotients of A.

The methods used here seem to be of some independent interest. For any algebra A whose complemented congruences form a Boolean sublattice of the congruence lattice of A, we construct an algebra $\mathfrak{R}A$ which lies in the variety generated by A. For a distributive lattice D, $\mathfrak{R}D$ is the injective hull of D. For rings, this construction is equivalent to taking the quotient ring with respect to a certain torsion theory, and has been investigated by Carson in [6]. In the special case of Boolean rings, $\mathfrak{R}R$ is the maximal ring of quotients of R (the analogue to Boolean rings of the MacNeille completion).

The paper has been divided into three sections. Section 1 gives the background on Pierce sheaves we require. In Section 2 we define the algebra $\mathfrak{R}A$ and discuss its properties in a general setting, and in Section 3 we apply our results to orthomodular lattices. For typographical reasons, the lattice operations of join and meet are written as $+$ and $\cdot$ respectively.

1. The Pierce Sheaf

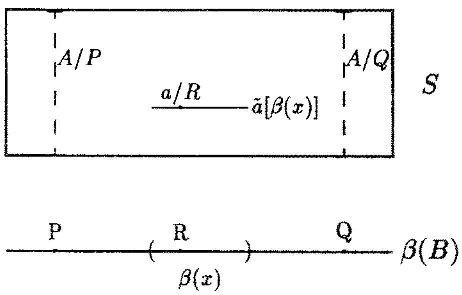
The basis of this paper is the Pierce sheaf representation of an algebra, see [5, 7, 8, 17, 20]. A description of the Pierce sheaf, and the properties we require, will comprise the remainder of this section.

DEFINITIONS. For an algebra A whose congruence lattice is distributive, the complemented elements of the congruence lattice of A form a Boolean sublattice B of the congruence lattice of A. The collection of prime ideals of B is denoted by $\beta(B)$. A topology is constructed on $\beta(B)$ from the basis of open sets $\{\beta(x) : x \in B\}$, where $\beta(x)$ is the set of all prime ideals of B containing x. The topological space $\beta(B)$ is customarily called the *Stone space* [5] of B. Note that for each point P in $\beta(B)$, P is an updirected family of congruences on A. Therefore $\bigcup P$ is a congruence on A which we will denote also by P. The system consisting of $\beta(B)$ and the indexed family of algebras $(A/P)_{P\in\beta(B)}$ is collectively referred to as the *Pierce sheaf* [20] of A.

It will save a good deal of essentially useless notation if we assume that A/P is disjoint from A/Q for distinct points P and Q of $\beta(B)$. Of course, this assumption is quite harmless. Setting $S = \cup\{A/P : P \in \beta(B)\}$, the usual Cartesian product is given by

$$\prod_{P\in\beta(B)} A/P = \{f : \beta(B) \to S : f(P) \text{ is in } A/P \text{ for each } P \in \beta(B)\}.$$

For an element a of A, we define a function $\tilde{a}$ in $\prod A/P$ by setting $\tilde{a}(P)$ equal to a/P for each point P of $\beta(B)$. We refer to such an $\tilde{a}$ as a *constant function*. The situation is depicted in the figure below.



In the following we assume that A is an algebra whose congruence lattice is distributive and B is the Boolean algebra of complemented congruences of A.

PROPOSITION 1. (Pierce).
1. $\{\tilde{a}[\beta(x)] : a \in A, x \in B\}$ *is a basis for a topology on* S.

2. *With respect to this topology, a function* $f \in \prod A/P$ *is continuous at a point* P *of* $\beta(B)$ *if and only if there is some open neighbourhood* $\beta(x)$ *of* P *and some* a *in* A *so that* f *agrees with* $\tilde{a}$ *on* $\beta(x)$.

3. $\{f \in \prod A/P : f$ *is continuous* $\}$ *is a subalgebra of* $\prod A/P$ *which we denote by* ΓA.

4. *For* $a, b \in A$ *and* $x \in B$, $\tilde{a}$ *agrees with* $\tilde{b}$ *on* $\beta(x)$ *if and only if* $(a, b) \in x$.

5. *The map* $\alpha : A \to \Gamma A$ *defined by* $a \rightsquigarrow \tilde{a}$ *is an embedding. It is an isomorphism if and only if the congruences in* B *are pairwise permuting.*

Proof. 1. If the point a/P is contained in the intersection of the basic open sets $\tilde{b}[\beta(x)]$ and $\tilde{c}[\beta(y)]$, then $P \in \beta(x) \cap \beta(y) = \beta(x + y)$ and $a/P = b/P = c/P$. So for some $z \in P$, (a, b) and (b, c) are elements of the congruence z, giving that a/P is an element of the basic open set $\tilde{a}[\beta(x + y + z)]$ which is contained in the intersection of $\tilde{b}[\beta(x)]$ and $\tilde{c}[\beta(y)]$.

2. From part 1, $\{\tilde{a}[\beta(x)] : x \in P\}$ is a neighbourhood basis for the point a/P. Therefore if f agrees with $\tilde{a}$ on $\beta(x)$ then f is continuous at each point of $\beta(x)$. But if f is continuous at the point P and $f(P)$ equals a/P, then there is a basic open neighbourhood $\beta(x)$ of P so that $f[\beta(x)]$ is contained in $\tilde{a}[\beta(B)]$. So f agrees with $\tilde{a}$ on $\beta(x)$.

3. To see that ΓA is a subalgebra of $\prod A/P$, suppose $f, g \in \Gamma A$ and $\oplus$ is a binary operation in the language of A. By part 2, for each point P of $\beta(B)$ there is an open neighbourhood N of P, and $a, b \in A$ so that f agrees with $\tilde{a}$ on N and g agrees with $\tilde{b}$ on N. Let $c = a \oplus b$, then as operations in $\prod A/P$ are componentwise, $f \oplus g$ agrees with $\tilde{c}$ on N. Therefore $f \oplus g$ is continuous at each point of $\beta(B)$. The generalization to operations depending on one or more variables is obvious, and

for a constant operation μ of A, the corresponding constant of ΓA is the function $\tilde{\mu}$.

4. If the pair (a, b) is in the congruence x, then as $x \in P$ for each point P of $\beta(x)$, we have that $\tilde{a}$ and $\tilde{b}$ agree on $\beta(x)$. Conversely, if (a, b) is not in x, then x is not in $\{w \in B : (a, b) \in w\} = F$. For x' the complement of x in B, we can extend $F \cup \{x'\}$ to an ultrafilter U over B. The complement of U in B is a prime ideal P of B which contains x. As $\tilde{a}(P)$ is not equal to $\tilde{b}(P)$, $\tilde{a}$ and $\tilde{b}$ do not agree on $\beta(x)$.

5. To check that α is well defined we must only check that $\tilde{a}$ is continuous for each $a \in A$, but this is shown in part 2. Also, α is a homomorphism since the operations in ΓA are componentwise and for each $P \in \beta(B)$ the natural map from A onto A/P is a homomorphism. That α is an embedding is a special case of part 4. If α maps A onto ΓA, we must show that the congruences in B are pairwise permuting. Equivalently, for congruences x, y in B, if (a, b) is in the congruence $x + y$ we must show there is some $c \in A$ with $(a, c) \in x$ and $(c, b) \in y$. If (a, b) is in $x + y$ then $\tilde{a}$ and $\tilde{b}$ agree on $\beta(x + y) = \beta(x) \cap \beta(y)$. Define $f \in \prod A/P$ so that f agrees with $\tilde{b}$ on $\beta(y)$ and agrees with $\tilde{a}$ otherwise. As $\beta(y)$ is clopen, f is continuous, so $f = \tilde{c}$ for some $c \in A$. Then as $\tilde{a}$ and $\tilde{c}$ agree on $\beta(x)$ and $\tilde{c}, \tilde{b}$ agree on $\beta(y)$, we have that (a, c) is in x and (c, b) is in y. Conversely, suppose the congruences in B permute. For $f \in \Gamma A$, using part 2 and the fact that $\beta(B)$ is compact, we can find $a_1, \ldots, a_n \in A$ and $z_1, \ldots, z_n \in B$ so that $\beta(z_1), \ldots, \beta(z_n)$ is an open cover of $\beta(B)$ and f agrees with $\tilde{a}_i$ on $\beta(z_i)$ for each $1 \leqslant i \leqslant n$. Then if $n \geqslant 2$, $\tilde{a}_1$ and $\tilde{a}_2$ agree on $\beta(z_1 + z_2)$ giving that (a_1, a_2) is in $z_1 + z_2$. As congruences in B permute, there is some $c \in A$ with (a_1, c) in z_1 and (c, a_2) in z_2. Then $\tilde{a}_1, \tilde{c}$ agree on $\beta(z_1)$ and $\tilde{c}, \tilde{a}_2$ agree on $\beta(z_2)$, giving that f agrees with $\tilde{c}$ on $\beta(z_1) \cup \beta(z_2)$. Repeating this procedure we see that $f = \tilde{a}$ for some $a \in A$.

2. The Algebra of Dense Open Sections

We introduce $\mathfrak{R}A$, the algebra of dense open sections of A, and describe some of its properties in a general setting.

DEFINITIONS. We will refer to a continuous function f in $\prod A/P$ as a *global section* of A and call ΓA the algebra of global sections of A. We say a function $f \in \prod A/P$ is a *dense open section* of A if the set of all points at which f is continuous contains a dense open subset of $\beta(B)$. As the collection of all dense open subsets of $\beta(B)$ is a filter base over $\beta(B)$, the set of all dense open sections is a subalgebra of $\prod A/P$ which we denote by $\Gamma_D A$. The filter base of dense open subsets of $\beta(B)$ naturally gives a congruence $\simeq$ over $\prod A/P$ where $f \simeq g$ if f and g agree on a dense open subset of $\beta(B)$. The quotient $(\Gamma_D A)/\simeq$ is the *algebra of dense open sections of* A and we denote it by $\mathfrak{R}A$. We say that the algebra A is *Hausdorff* if $S = \bigcup A/P$ is a Hausdorff space, and A is *weakly Hausdorff* if the natural map $A \to \mathfrak{R}A$ is an embedding.

NOTATION. For functions f, g from a topological space X to a topological space Y, let Cf be the set of points at which f is continuous, and let $[\![f = g]\!]$ be the set of points at which f and g agree. Note that for $f \in \prod A/P$, part 2 of Proposition 1 gives that Cf is open in $\beta(B)$. Finally, we let Δ denote the smallest congruence on A. Note that $\beta(\Delta)$ contains all points in $\beta(B)$.

PROPOSITION 2.
1. *The following are equivalent,*
(i) *A is Hausdorff.*
(ii) *For each $a, b \in A$, $[\![\tilde{a} = \tilde{b}]\!]$ is clopen.*
(iii) *For each $a, b \in A$ there is a least congruence in B which contains (a, b).*
2. *The following are equivalent and are implied by each of the above conditions*
(iv) *A is weakly Hausdorff.*
(v) *If $T \subseteq B$ and Δ is the meet of T in B, then Δ is the meet of T in the congruence lattice of A.*
(vi) *All existing meets in B agree with those in the congruence lattice of A.*
(vii) *The natural map $\Gamma A \to \mathfrak{R}A$ is an embedding.*
(viii) *If two dense open sections of A agree on a dense open set, then they agree at every point where both are continuous.*

Proof. 1. To see that the first condition implies the second, note that $[\![\tilde{a} = \tilde{b}]\!] = \tilde{b}^{-1}[\tilde{a}[\beta(\Delta)]]$ is open since $\tilde{b}$ is continuous. But $[\![\tilde{a} = \tilde{b}]\!]$ is closed if S is Hausdorff.

To see that the second condition implies the third, for $a, b \in A$ we have by part 4 of Proposition 1 that (a, b) is in z if and only if $[\![\tilde{a} = \tilde{b}]\!] \supseteq \beta(z)$. As $[\![\tilde{a} = \tilde{b}]\!]$ is clopen, $[\![\tilde{a} = \tilde{b}]\!] = \beta(w)$ for some $w \in B$. Then w is the least member of B which contains (a, b).

To see that the third condition implies the first, we must show that any two distinct points in S can be separated by disjoint open neighbourhoods. If P and Q are distinct, obviously a/P and b/Q can be separated since $\beta(B)$ is Hausdorff. If $a, b \in A$ and a/P and b/P are distinct, let z be the least member of B which contains (a, b). For z' the complement of z in B, as z is not in P we have that z' is in P. Then a/P is in $\tilde{a}[\beta(z')]$ and b/P is in $\tilde{b}[\beta(z')]$. We have only to show that $\tilde{a}[\beta(z')]$ and $\tilde{b}[\beta(z')]$ are disjoint. If c/Q is a point in their intersection, then $a/Q = b/Q$ implying that z is in Q, an impossibility.

2. To see that the fourth condition implies the fifth, suppose that T is a subset of B and Δ is the meet of T in B. Then $E = \bigcup\{\beta(x) : x \in T\}$ is a dense open set. If $(a, b) \in \bigcap T$, then $\tilde{a}$ and $\tilde{b}$ agree on E. So if A is weakly Hausdorff, then $a = b$.

To see that the fifth condition implies the sixth, suppose that T is a subset of B and z is the meet of T in B. For z' the complement of z in B, Δ is the meet of $T \cup \{z'\}$ in B. By our assumption $\bigcap T \cap z'$ is also equal to Δ, and from general considerations $\bigcap T$ contains z. Using the modular law and the fact that z and z' are complements we have that $\bigcap T$ is equal to z.

To see that the sixth condition implies the seventh, suppose that f and g are global sections of A, and $f \simeq g$. As f and g are continuous, for a point P in $\beta(B)$ there is a basic open neighbourhood $\beta(z)$ of P and $a, b \in A$ so that f agrees with $\tilde{a}$ on $\beta(z)$ and g agrees with $\tilde{b}$ on $\beta(z)$. As f and g agree on a dense open set, $E = [\![f = g]\!] \cap \beta(z)$ contains an open set which is dense in $\beta(z)$. So for $T = \{x \in B : \beta(x) \subseteq E\}$, z is the meet of T in B. But $\tilde{a}$ and $\tilde{b}$ agree on $\beta(x)$ for each $x \in T$, so (a, b) is in $\bigcap T$. Therefore (a, b) is in z, giving $f(P) = g(P)$.

To see that the seventh condition implies the eighth, let f and g be dense open sections of A which agree on a dense subset of $\beta(B)$. If P is a point of continuity of both f and g, then by part 2 of Proposition 1 there is a basic open neighbourhood $\beta(x)$ of P and elements a, b in A so that f agrees with $\tilde{a}$ on $\beta(x)$ and g agrees with $\tilde{b}$ on $\beta(x)$. Define h in $\prod A/P$ so that h agrees with $\tilde{b}$ on $\beta(x)$ and h agrees with $\tilde{a}$ otherwise. As $\beta(x)$ is clopen h is a global section of A. But h agrees with $\tilde{a}$ everywhere that f agrees with g, so $h \simeq \tilde{a}$. As the map $\Gamma A \to \mathfrak{R}A$ is an embedding, h equals $\tilde{a}$, so $f(P) = g(P)$.

It is obvious that the eighth condition implies the fourth, since $\tilde{a}$ is continuous for each a in A.

That the first condition implies the fourth follows from the general fact that continuous maps into a Hausdorff space which agree on a dense set are equal.

The equivalence of conditions (i) and (ii) was first shown in [17].

REMARK. For any algebra A, if B is some collection of complemented congruences of A which form a Boolean sublattice of the congruence lattice of A, then one could proceed as above to produce a sheaf for A over the Stone space of B. Proposition 1 remains valid in this more general setting as does the first part of Proposition 2. The second part of Proposition 2 seems to require the modularity of the congruence lattice of A.

PROPOSITION 3. *If A is a weakly Hausdorff algebra which generates a congruence distributive variety, then the natural embedding $\Gamma A \to \mathfrak{R}A$ is an essential extension.*

Proof. We must show that every nontrivial congruence of $\mathfrak{R}A$ restricts to a nontrivial congruence of ΓA. Suppose that θ is a nontrivial congruence on $\mathfrak{R}A$, then there are f and g in $\Gamma_D A$ with $(f/\simeq, g/\simeq)$ in θ and $\not\simeq$ g. So by part (viii) of Proposition 2 there is a point P at which both f and g are continuous and $f(P)$ is not equal to $g(P)$. Then there is a clopen neighbourhood K of P and elements a and b of A so that f agrees with $\tilde{a}$ on K and g agrees with $\tilde{b}$ on K. Define h in $\prod A/P$ so that h agrees with $\tilde{a}$ on K and h agrees with $\tilde{b}$ otherwise. Then h is in ΓA. We will show that $(h/\simeq, \tilde{b}/\simeq)$ is in θ. Let

$$\lambda = \{(p/\simeq, q/\simeq) : p \text{ and } q \text{ agree on a dense open subset of } K\}$$
$$\phi = \{(p/\simeq, q/\simeq) : p \text{ and } q \text{ agree on a dense open subset of } \neg K\}.$$

Then λ and ϕ are congruences on $\mathfrak{R}A$ and as A is weakly Hausdorff $\lambda \cdot \phi = \Delta$. As A generates a congruence distributive variety, $\theta = (\theta + \lambda) \cdot (\theta + \phi)$. Obviously $(h/\simeq, \tilde{b}/\simeq)$ is in ϕ so we need only show that $(h/\simeq, \tilde{b}/\simeq)$ is in $\theta + \lambda$. But $(h/\simeq, f/\simeq)$ and $(g/\simeq, b/\simeq)$ are in λ and $(f/\simeq, g/\simeq)$ is in θ, so $(h/\simeq, b/\simeq)$ is in $\theta + \lambda$.

PROPOSITION 4. *Let A be a weakly Hausdorff algebra which has a bounded lattice as a reduct, and let B be the Boolean algebra of complemented congruences of A. If there is a dense open subset G of $\beta(B)$ and a natural number n so that for each point P in G every chain in A/P has at most n elements, then $\mathfrak{R}A$ is the MacNeille completion of ΓA.*

Proof. As A has a bounded lattice as a reduct, the congruence lattice of A is distributive ([5], p. 80), and as ΓA and $\mathfrak{R}A$ are in the variety generated by A, ΓA and $\mathfrak{R}A$ also have bounded lattices as reducts. As A is weakly Hausdorff, by Proposition 2 there is a natural embedding of ΓA into $\mathfrak{R}A$, so the lattice reduct of ΓA is a sublattice of the lattice reduct of $\mathfrak{R}A$. We must show [3] that the lattice reduct of $\mathfrak{R}A$ is complete and that every element of $\mathfrak{R}A$ is both the join and meet (of images) of elements of ΓA. To show that this lattice is complete, it is enough to show that for any nonempty set T of dense open sections of A, $\{f/\simeq : f \in T\}$ has a supremum in $\mathfrak{R}A$. Let C be the set of all points of G at which *some* member of T is continuous. Choose g in $\prod A/P$ so that $g(P) = \sum\{f(P) : f \in T \text{ and } P \in Cf\}$ for each P in C.

CLAIM. *g is a dense open section of A.*

Proof. For each nonempty open set N of $\beta(B)$, we must show g is continuous at some point of N. Consider a tower of nonempty open sets $N \supseteq M_1 \supseteq \cdots \supseteq M_k$ which satisfies the following conditions

- for each $1 \leqslant i \leqslant k$ there is $f_i \in T$ and $a_i \in A$ so that f_i agrees with $\tilde{a}_i$ on M_i
- for each point P in M_k, $f_1(P), f_1(P) + f_2(P), \ldots, \sum_{i=1}^{k} f_i(P)$ is a strictly increasing chain in A/P.

Clearly there is at least one such tower, since T is nonempty and a member of T is dense open section of A. However, any such tower can be of length at most n, since each nonempty open set intersects G nontrivially. Therefore we can choose such a tower $N \supseteq M_1 \supseteq \cdots \supseteq M_q$ of maximal length, with $f_1, \ldots, f_q$ in T and $a_1, \ldots, a_q$ in A satisfying the above conditions.

Let $a = \sum_{i=1}^{q} a_i$, we wish to show that g agrees with $\tilde{a}$ on M_q thereby proving our claim. Suppose that Q is a point in M_q and g does not agree with $\tilde{a}$ at Q. Since f_i is continuous at Q for each $1 \leqslant i \leqslant q$, we have that $\tilde{a}(Q) \leqslant g(Q)$, so there must be some f_{q+1} in T which is continuous at Q with $f_{q+1}(Q) \nleqslant \tilde{a}(Q)$. By part 2 of Proposition 1 we can find an open neighbourhood $\beta(x) \subseteq M_q$ of Q and an element a_{q+1} in A so that f_{q+1} agrees with $\tilde{a}_{q+1}$ on $\beta(x)$. Set $b = \sum_{i=1}^{q+1} a_i$ and define h in $\prod A/P$ so that h agrees with $\tilde{b}$ on $\beta(x)$ and h agrees with $\tilde{a}$ otherwise.

As $\beta(x)$ is clopen, h is a global section of A. But h does not agree with $\tilde{a}$ at Q, and as A is weakly Hausdorff h can not agree with $\tilde{a}$ on a dense open set. The set where h agrees with $\tilde{a}$ is given by $h^{-1}[\tilde{a}[\beta(\Delta)]]$ which is open, so there is a nonempty open set M_{q+1} on which h and $\tilde{a}$ differ. From the definition of h, M_{q+1} is contained in $\beta(x)$, which in turn is contained in M_q. Then for each P in M_{q+1}, $f_1(P), \ldots, \sum_{i=1}^{q+1} f_i(P)$ is a strictly increasing chain in A/P. This contradicts our choice of a tower of maximal length establishing our claim.

For each f in T and each point P at which both f and g are continuous we have $f(P) \leqslant g(P)$. Therefore $(f+g) \simeq g$ and as $\simeq$ is a congruence $f/\simeq + g/\simeq$ is equal to $g/\simeq$ giving that $f/\simeq \leqslant g/\simeq$. So $g/\simeq$ is an upper bound of $\{f/\simeq : f \in T\}$. Suppose that h is a dense open section of A and $h/\simeq$ is an upper bound of $\{f/\simeq : f \in T\}$. Then for each f in T, $(f+h) \simeq h$ so by part (viii) of Proposition 2, $(f+h)$ agrees with h at each point where both are continuous. So $f(P) \leqslant h(P)$ for each point P at which f and h are both continuous. If P is a point in C at which h is continuous then $g(P) \leqslant h(P)$. But C and Ch are dense open sets, so $(g+h) \simeq h$ giving $g/\simeq \leqslant h/\simeq$. Therefore $g/\simeq$ is the least upper bound of $\{f/\simeq : f \in T\}$.

We are left to show that each element of $\mathfrak{R}A$ is the join and meet (of images) of elements of ΓA. Let h be a dense open section of A. For each point P at which h is continuous we can find a clopen neighbourhood $\beta(x_P)$ of P and an element a_P in A so that h agrees with $\tilde{a}_P$ on $\beta(x_P)$. Let f_P be the global section of A which agrees with $\tilde{a}_P$ on $\beta(x_P)$ and agrees with $\tilde{0}$ otherwise, where $\tilde{0}$ is the zero of the lattice reduct of A. Using an argument similar to that of the preceding paragraph it is easily seen that h is the least upper bound of $\{f_P/\simeq : P \in Ch\}$.

3. Applications to Orthomodular Lattices

An ortholattice is a bounded lattice $(L, +, \cdot, 0, 1)$ with an additional unary operation $'$ which is an order inverting complementation of period two. An orthomodular lattice is an ortholattice which satisfies the following condition known as the orthomodular law;

for all x, y in L, if $x \leqslant y$ then $x + (x' \cdot y) = y$.

It is not difficult to see that orthomodular lattices can be defined by equations, and therefore form a variety of algebras.

An element c of a bounded lattice L is called *central* if c has a complement in L and for each $a, b \in L$, the sublattice of L generated by $\{a, b, c\}$ is distributive. It is easily seen that the collection of all central elements of a lattice L forms a Boolean sublattice of L. An element c of an orthomodular lattice A is called central if c is central in the lattice reduct of A. For an orthomodular lattice A and c in the centre of A define

$$\gamma(c) = \{(a, b) \in A^2 : a \cdot c' = b \cdot c'\}.$$

It is shown in ([16], pp. 73–80) that $\gamma(c)$ is a congruence of A, and in fact γ is an isomorphism between the Boolean algebra of central elements of A and the Boolean algebra of complemented congruences of A. It follows that an orthomodular lattice is directly irreducible if and only if its centre consists just of the bounds 0 and 1.

DEFINITIONS. Let S be a set of directly irreducible orthomodular lattices, and φ be a first order formula in the language of ortholattices. We say that a term t *returns the truth value of* φ *for* S if for each L in S and each $\mathbf{x}$ in L,

$$t(\mathbf{x}) = \begin{cases} 1 & \text{if } L \models \varphi(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

We say that a term $p(\mathbf{x}, y)$ *returns least central upper bounds for* φ *and* S if for each L in S and each $\mathbf{x}$ in L with $L \models \varphi(\mathbf{x})$

$$p(\mathbf{x}, y) = \begin{cases} 1 & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Such a term $p(\mathbf{x}, y)$ is said to be *consistent* if for any orthomodular lattice L and any $\mathbf{x}$ in L

$$p(\mathbf{x}, y) = y \quad \text{if } y \text{ is central in } L.$$

The following two theorems were proved in [12].

THEOREM 1. *Let S be a set of directly irreducible orthomodular lattices with a finite uniform upper bound on the lengths of their chains. Then there is a first order formula φ, satisfiable in some member of S, and terms $t(\mathbf{x})$ and $p(\mathbf{x}, y)$ so that t returns the truth value of φ for S, while p is consistent and returns least central upper bounds for φ and S.*

THEOREM 2. *Let S be a set of orthomodular lattices having a finite uniform upper bound n on the lengths of their chains. Then any directly irreducible algebra A in the variety generated by S is simple, and each chain in A has at most n elements.*

PROPOSITION 5. *Let A be an algebra in a variety generated by a set of orthomodular lattices having a finite uniform upper bound on the lengths of their chains, and let B be the Boolean algebra of complemented congruences of A. Then*

1. *A is weakly Hausdorff.*
2. *$\{P \in \beta(B)$: each chain in A/P has at most n elements $\}$ contains a dense open set.*

Proof. Let $\mathcal{K}$ be the set of orthomodular lattices generating our variety V. As every orthomodular lattice has a lattice reduct, the congruence lattice of an orthomodular lattice is distributive. So by Jónsson's theorem [15], the subdirectly irreducibles in V are in $HSP_u(\mathcal{K})$. Then by Łoś' theorem ([5], p. 280) each chain in a subdirectly irreducible in V has at most n elements. By Birkhoff's theorem ([5], p. 58), A is isomorphic to subdirect product of a family $(A_i)_I$ of subdirectly irreducibles in V. For convenience, we identify A with its image in $\prod A_i$. Note that as each A_i is subdirectly irreducible, and therefore directly irreducible, Theorem 2 implies that a chain in A_i can have at most n elements. The following simple observation will be of great use;

c is in the centre of A if and only if $c(i) \in \{0, 1\}$ for each $i \in I$.

1. To show that A is weakly Hausdorff, it is enough to verify the equivalent statement given in part (v) of Proposition 2. Suppose that T is a subset of B, and the meet of T in the congruence lattice of A is not equal to Δ. We must show that the meet of T in B is not equal to Δ. As congruences in an orthomodular lattice are determined by one of their equivalence classes ([16], pp. 73–80), there is some nonzero a in A with $(0, a)$ in $\bigcap T$. Let U be the set of all central elements c of A such that $\gamma(c)$ is in T. As $(0, a)$ is in $\gamma(c)$ for each c in U, it follows that a is a lower bound of U.

Having seen that each A_i is directly irreducible, and that a chain in A_i has at most n elements, we may apply Theorem 1 to the set $S = \{A_i : a(i) \neq 0\}$. Let φ, t, and p be the resulting formula and terms. As φ is satisfiable in some member of S, we can find some $\mathbf{x}$ in A so that $A_j \models \varphi(\mathbf{x}(j))$ for some A_j in S. Let b be the element of A given by

$$b = t(\mathbf{x}) \cdot p(\mathbf{x}, a).$$

As $b(i)$ is either 0 or 1 for each i in I, b is in the centre of A. Also, $b(j) = 1$, since $A_j \models \varphi(\mathbf{x}(j))$. And for each i in I with $b(i)$ equal to 1, $a(i)$ is nonzero. As a is a lower bound of U, and U is contained in the centre of A, b is also a lower bound of U. Therefore the meet of U in the centre of A is not equal to 0, and as γ is an isomorphism, the meet of T in B is not equal to Δ.

2. By Theorem 2, it is enough to show that the set of all points P in $\beta(B)$, for which A/P is directly irreducible, contains a dense open set. This is equivalent to showing that beneath each nonzero central element c of A there is a nonzero central element d of A with A/P directly irreducible for each P in $\beta(d')$.

Let c be a nonzero central element of A. As before, we may apply Theorem 1 to the set $S = \{A_i : c(i) = 1\}$. Let φ, t, and p be the resulting formula and terms. As φ is satisfiable in some member of S, we can find $\mathbf{x}$ in A so that $A_j \models \varphi(\mathbf{x}(j))$ for some A_j in S. Let d be the element of A defined by

$$d = c \cdot t(\mathbf{x}).$$

Then d is a nonzero element in the centre of A and further, for each a in A

$p(\mathbf{x}, a \cdot d)$ is in the centre of A.

We have only to show that A/P is directly irreducible for each P in $\beta(d')$. Note that for any e in the centre of A,

$$e/P = \begin{cases} 0/P & \text{if } \gamma(e) \text{ is in } P \\ 1/P & \text{otherwise.} \end{cases}$$

In particular, $d/P = 1/P$.If a/P is in the centre of A/P, then as p is consistent

$$a/P = p(\mathbf{x}/P, a/P) = p(\mathbf{x}, a \cdot d)/P.$$

So the centre of A/P consists of just $0/P$ and $1/P$, giving that A/P is directly irreducible.

THEOREM 3. *If V is a variety generated by a set of orthomodular lattices having a finite uniform upper bound on the lengths of their chains, then for each A in V, $\mathfrak{R}A$ is the MacNeille completion of A and $\mathfrak{R}A$ is in V.*

Proof. It is well known that all orthomodular lattices have permuting congruences ([16], p. 83), so the result follows by part 5 of Proposition 1, Proposition 4 and Proposition 5.

While Theorem 3 gives sufficient conditions for a variety of orthomodular lattices to be closed under MacNeille completions, these conditions are not necessary. Let V be the variety of orthomodular lattices defined by the identity $\gamma(x, \gamma(y, z)) \approx 0$ where $\gamma(x, y)$ is defined by

$$\gamma(x, y) = (x + y) \cdot (x + y') \cdot (x' + y) \cdot (x' + y').$$

The term $\gamma(x, y)$ is usually referred to as the *commutator* of x and y. It can be shown that V is closed under MacNeille completions, but is not generated by any set of orthomodular lattices having a finite uniform upper bound on the lengths of their chains. The proof of this fact is non-trivial. It is based upon properties of the algebra $\mathfrak{R}A$ developed in [6], and the observation that each algebra in V is weakly Hausdorff.

An orthomodular lattice A is said to have n commutators if there are n elements in the image of the map $\gamma : A^2 \to A$. The variety of Boolean algebras is generated by the class of all orthomodular lattices having one commutator, while the variety V above is generated by the class of all orthomodular lattices having at most two commutators. This suggests the following question.

PROBLEM. If V is generated by the class of all orthomodular lattices having at most n commutators, is V closed under MacNeille completions ?

An encouraging step towards a positive solution to this problem is given in a recent result of Greechie [9], which states that the MacNeille completion of a commutator finite orthomodular lattice is a commutator finite orthomodular lattice.

Acknowledgement

I would like to thank the referee for carefully reading the manuscript and making several helpful comments, particularly for shortening the proof of Proposition 3.

References

1. D. H. Adams (1969) The completion by cuts of an orthocomplemented modular lattice, *Bull. Austral. Math. Soc.* **1**, 279–280.
2. D. R. Balbes and P. Dwinger (1974) *Distributive Lattices*, University of Missouri Press.
3. B. Banaschewski (1956) Hullensysteme und Erweiterungen von Quasi-Ordnungen, *Z. Math. Logik Grundl. Math.* **2**, 35–46.
4. G. Bruns, R. J. Greechie, J. Harding, and M. Roddy (1990) Completions of orthomodular lattices, *Order* **7**, 67–76.
5. S. Burris and H. P. Sankappanavar (1981) *A Course in Universal Algebra*, Springer.
6. A. Carson (1989) *Model Completions, Ring Representations and the Topology of the Pierce Sheaf*, Pitman research notes in mathematics series 209.
7. S. D. Comer (1971) Representations by algebras of sections over Boolean spaces, *Pacific J. of Math.* **38**, 29–38.
8. B. A. Davey (1973) Sheaf spaces and sheaves of universal algebras, *Math. Zeit.* **134**, 275–290.
9. R. J. Greechie, private communication.
10. P. R. Halmos (1967) *Lectures on Boolean Algebras*, Van Nostrand.
11. J. Harding (1991) Orthomodular lattices whose MacNeille completions are not orthomodular, *Order* **8**, 93–103.
12. J. Harding (1992) Irreducible orthomodular lattices which are simple, *Algebra Universalis* **29**, 556–563.
13. J. Harding (1994) Any lattice can be embedded into the MacNeille completion of a distributive lattice (to appear in *The Houston Journal of Math.*).
14. P. T. Johnstone (1982) *Stone Spaces*, Cambridge University Press.
15. B. Jónsson (1967) Algebras whose congruence lattices are distributive, *Math. Scand.* **21**, 110–121.
16. G. Kalmbach (1983) *Orthomodular Lattices*, Academic Press.
17. A. Macintyre (1973) Model-completeness for sheaves of structures, *Fund. Math.* **81**, 73–89.
18. M. D. MacLaren (1964) Atomic orthocomplemented lattices, *Pac. J. Math.* **14**, 597–612.
19. H. M. MacNeille (1937) Partially ordered sets, *Trans. AMS* **42**, 416–460.
20. R. S. Pierce (1967) Modules over commutative regular rings, *Mem. Amer. Math. Soc.* **70**.
21. J. Schmidt (1956) Zur Kennzeichnung der Dedekind MacNeilleschen Hulle einer geordneten Menge, *Archiv d. Math.* **7**, 241–149.