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A Bundle Representation for Continuous Geometries

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We show that a reducible continuous geometry can be represented as the continuous sections of a bundle of irreducible continuous geometries. We relate this bundle representation to the Pierce sheaf of the continuous geometry and to the subdirect product representation developed by Maeda. © 1997 Academic Press

1. INTRODUCTION

In the 1930s, von Neumann introduced the notion of a *continuous geometry* to serve as a point-free generalization of projective geometry. He defined a continuous geometry to be a complete, irreducible, complemented, modular lattice which is both join and meet continuous. This definition was no doubt motivated by the work of Birkhoff and Menger, who showed that the irreducible, complemented, modular lattices of height n were exactly those that arose as the subspace lattices of $n-1$-dimensional projective geometries. In his remarkable paper [7], von Neumann showed that a continuous geometry L has a dimension function D: $L \to [0,1]$ satisfying among other properties that (i) $D(a) + D(b) = D(a \vee b) + D(a \wedge b)$, (ii) $D(0) = 0$, and (iii) $D(1) = 1$. In the case that L is the lattice of subspaces of an $n-1$-dimensional projective geometry, this

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dimension function is just the normalized dimension of the subspace, taking the values $\{i/n: i = 0, 1, \ldots, n\}$.

von Neumann intended to continue his study by removing the irreducibility requirement from the definition of a continuous geometry. He never completed this task and the job was taken over by several Japanese mathematicians including Iwamura, Kawada *et al.*, and Maeda. Their work has become best known through Maeda's book [5], and we shall collectively refer to their results as Maeda's. They were able to show that a continuous geometry has a dimension function in a more general sense (which we describe in detail in our Section 4) and as a by-product of this result they obtained that any continuous geometry could be represented as a subdirect product of irreducible continuous geometries. Much later, Nishimura [6] used techniques from Boolean valued set theory to show the existence of this generalized dimension function was a direct consequence of von Neumann's result for the irreducible case.

In this paper we extend the results of Maeda to show that any continuous geometry can be represented as the continuous sections of a bundle whose stalks are all irreducible continuous geometries. The topology of this bundle is such that the subspace topology on each stalk is the usual metric space topology of an irreducible continuous geometry. Further, the dimension functions on the stalks form a continuous map from the bundle into the reals. We feel that this result is a natural extension of Maeda's subdirect product representation and is helpful in explaining the nature of Maeda's generalized dimension function. The similarity of this result to the classical representations of von Neumann algebras as rings of continuous functions is striking and points out the amazing connections between the seemingly algebraic definition of a continuous geometry and its analytical consequences.

Any study of the generalized dimension function of a continuous geometry is necessarily tied to the Pierce sheaf of a continuous geometry. That Maeda's work predated the introduction of the Pierce sheaf does not lessen the connection. In fact, it is remarkable the degree to which Maeda seems to have anticipated results from the theory of Pierce sheaves, particularly with his definition of Z-lattices and the attention paid to equalizers of first order formulas. Nishimura's work is also closely linked to Pierce sheaves. His result is obtained using a meta-theorem about Pierce sheaves combined with a description of the real numbers in a Boolean valued set theory. We have tried to pay particular attention to the connection between the Pierce sheaf of a continuous geometry and its bundle representation. This approach has allowed us to give a concrete description, in lattice theoretic terms, of the bundle associated with a continuous geometry.

The paper is organized in the following fashion. In the second section we provide some background on bundles and Pierce sheaves, and we give a thorough account of the properties of the Pierce sheaf of a continuous geometry. In the third section we describe how the Pierce sheaf of a continuous geometry is related to the subdirect product representation given by Maeda. In the fourth and final section we prove our main results about the bundle representation, and we describe the connections between this bundle and the Pierce sheaf.

Whenever possible we have followed the notation and terminology of Maeda [5]. In particular, L will denote a continuous geometry in the general sense, Z will denote the center of L, and X will denote the Stone space (see [1]) of Z. For background on general lattice theory, consult [2].

2. THE PIERCE SHEAF

A *bundle* is a triple (S, X, π), where S and X are topological spaces and $\pi\colon S \to X$ is a continuous map onto X. A *sheaf* is a bundle for which π is a local homeomorphism, i.e., there is a neighborhood of each point of S on which π is a homeomorphism. For each $x \in X$, we call $S_x = \pi^{-1}[x]$ the *stalk* over x. Note that for a sheaf, the subspace topology on a stalk is discrete; the same is not generally true of bundles. A *section* is a map σ: $X \to S$ such that $\pi \circ \sigma = \mathrm{id}_X$, i.e., an element of the product $\prod_X S_x$. The only sections which we shall consider are the continuous ones.

There is a sheaf naturally associated with each bounded lattice—the *Pierce sheaf*. We will describe the Pierce sheaf of our continuous geometry L. Each prime ideal $P \in X$ of the center of L induces a congruence on L with a being related to b if $a \wedge e = b \wedge e$ for some central $e \notin P$. Let S denote the disjoint union of the lattices L/P, and for each $a \in L$ and each clopen subset $K \subseteq X$ define $B(a, K) = \{a/P\colon\ P \in K\}$. The sets of the form $B(a, K)$ form a basis for a topology on S and in fact (S, X, π) is a sheaf of lattices. The utility of this sheaf is provided by the fact that L is isomorphic to the lattice of continuous sections of (S, X, π). Each $a \in L$ defines a continuous section $\tilde{a}$, where $\tilde{a}(P) = a/P$, and every continuous section arises in this manner.

The Pierce sheaf of an arbitrary lattice is usually not particularly well behaved. However, by virtue of the fact that a continuous geometry is a Z-lattice [5], we can do much better [3]. We know that the space S is Hausdorff and that the stalks L/P are all directly irreducible. However, to describe the most remarkable property of this sheaf we must first define the notion of an equalizer. For any n-tuple $\mathbf{a}$ of elements of L and any first order formula φ, the equalizer $[[\varphi(\mathbf{a})]]$ is defined to be $\{P \in X$:

$L/P \vDash \varphi(\mathbf{a}/P)\}$. The amazing fact [3] is that for the Pierce sheaf of a Z-lattice, the equalizer of *any* first order formula is clopen.

These facts would doubtless have been of great use to Maeda in his construction of a dimension function on L. However, little is to be gained in trying to use these techniques to shorten Maeda's proof. Nishimura [6] used the Pierce sheaf representation and a meta-theorem from Boolean valued set theory to show that the existence of a dimension function in the general case is an immediate consequence of von Neumann's proof of the existence of a dimension function in the irreducible case.

To complete our description of the Pierce sheaf of L, we must gain a better knowledge of the stalks L/P. As we shall see, the stalks L/P are quite well behaved. Aside from the obvious properties that they are complemented modular lattices, they also satisfy the following:

(TP) Transitivity of perspectivity.

(GC) Generalized comparability. For $a, b \in L$, either a is perspective to a subelement of b or b is perspective to a subelement of a.

That the L/P satisfy (TP) follows from the general fact that any homomorphic image of a complemented modular lattice satisfying (TP) must satisfy (TP). That the L/P satisfy (GC) follows from [5, Satz 1.1, p. 87].

Of equal importance to the properties that the L/P do have are the properties which they do not have. Before we can explain further, we need a bit of notation. Let M be any complemented modular lattice satisfying (TP). We let $[a]$ denote the equivalence class of a under perspectivity, and say that $[a] + [b]$ is defined if there are $a_1 \in [a]$ and $b_1 \in [b]$ such that $a_1 \wedge b_1 = 0$. In this case, we set $[a] + [b] = [a_1 \vee b_1]$. For a natural number n, the shorthand $n[a]$ has the obvious meaning. For a discussion of the properties of this operation, see [5, pp. 97–99]. We are interested then in the property

(IP) Interpolation property. If $n[a]$ is defined for all $n \in \mathsf{N}$, then $a = 0$.

It follows from [5, Satz 4.3, p. 100] that any continuous geometry satisfies (IP). However, as the following example will show, the stalks L/P are not continuous geometries because they are not in general complete and need not satisfy (IP). In many ways, it is the condition (IP) that lies at the heart of matters as we shall see in the next section.

EXAMPLE 2.1. For each natural number n let L_n be the lattice of subspaces of an n-dimensional vector space over the reals. Then each L_n is a complete complemented continuous modular lattice, and hence the same can be said of the product $L = \Pi_n L_n$. The center Z of L is obviously isomorphic to the power set of the natural numbers. Further, there is a bijection between points of the Stone space X of the center of L and the collection of all ultrafilters over the natural numbers. By a careful examination of the definitions involved, one can verify that for a prime ideal P of the center of L, the lattice L/P is isomorphic to the ultraproduct $\Pi_n L_n/\mathsf{U}$, where U is the ultrafilter over the natural numbers associated with P.

The foregoing argument has shown that up to isomorphism, the family of lattices $\{L/P\colon P \in X\}$ is just the family of all ultraproducts of the L_n taken over the natural numbers. If P is a principal prime ideal, then L/P is a principal ultraproduct and hence isomorphic to some L_n. Thus, for a principal prime ideal P, the lattice L/P is complete and satisfies (IP). However, if the prime ideal P is nonprincipal, then L/P corresponds to a nonprincipal ultraproduct of the lattices L_n. Using standard arguments about ultraproducts, one can show that for any nonprincipal prime ideal P, the lattice L/P is not complete and does not satisfy (IP).

3. SUBDIRECT PRODUCT REPRESENTATIONS

Let Y be the set of maximal neutral ideals (p-ideals) of L. Maeda shows [5, Satz 3.1, p. 123] that there is a natural bijection of Y onto X, and he indeed often identifies the two sets. He also shows [5, Hilfsatz 3.3, p. 124] that for each $J \in Y$, L/J is a simple continuous geometry, and that the intersection of the maximal neutral ideals is trivial. Therefore L may be represented as a subdirect product of the lattices $\{L/J\colon J \in Y\}$.

The focus of this section will be the nature of the bijection between Y and X. If J is a maximal neutral ideal of L, then the associated prime ideal of Z is simply $J \cap Z$. Conversely, for any prime ideal of Z, we agree to let J_P denote the unique maximal neutral ideal having the property that $P = J_P \cap Z$. Of course Maeda established the nature of J_P via the dimension function D [5, Satz 3.1, p. 123], while in [4], J_P is characterized in terms of a family of central elements that were used in constructing D. We proceed here to give a direct lattice theoretic characterization of the J_P and their connection to the stalks L/P of the Pierce sheaf. As we shall soon see, the key to all of this is given by the condition we called (IP).

For the moment, we shall be working in a complemented modular lattice M that satisfies (GC) and (TP). We agree to let

$$K = \{a \in M\colon n[a] \text{ exists for all } n \in \mathsf{N}\}.$$

It turns out that K is the unique maximal neutral ideal of M.

LEMMA 3.1. *Let J be a neutral of M. Then*:

(1) *If $a \in J$, then $[a] \subseteq J$.*

(2) *If $a, b \in J$ and $[a] + [b]$ is defined, then $[a] + [b] \subseteq J$.*

(3) *If $a \in J$ and $n[a]$ is defined, then $n[a] \subseteq J$.*

(4) *If $a \in J$ and $[b] \preccurlyeq [a]$, then $[b] \subseteq J$.*

Proof. Assertion (1) follows from J being closed under perspectivity, while (2) follows from (1) and the definition of $[a] + [b]$. (3) follows from (2) by induction, so we need only consider (4). To say that $[b] \preccurlyeq [a]$ is to say that b is perspective to some $a_1 \leq a$, but now $a_1 \in J$ since J is an ideal, and by (1), $[b] \subseteq J$. ∎

THEOREM 3.2. *Every proper neutral ideal J of M is contained in K.*

Proof. The proof will be by contradiction. Suppose $a \in J$, $n[a]$ is defined, but $(n+1)[a]$ is not defined. By Lemma 3.1, if $n[a] = [b]$, then $b \in J$. Since $(n+1)[a]$ is not defined, $2[b]$ is not defined. However, now let b' denote a complement of b. By (GC), we must have $[b] \preccurlyeq [b']$ or $[b'] \preccurlyeq [b]$. Since $[b] \preccurlyeq [b']$ would force $2[b]$ to exist, it follows that $[b'] \preccurlyeq [b]$. However, then by Lemma 3.1, $b' \in J$, so $1 = b \vee b' \in J$, contrary to J being proper. ∎

We are now ready to prove the result that we need.

THEOREM 3.3. *K is the unique maximal neutral ideal of M.*

Proof. In view of Theorem 3.2, we would be done if we could just show that K is a neutral ideal. Since it is clearly an order ideal that is closed under perspectivity, it suffices to prove that it is closed under finite joins.

Let $a, b \in K$. If necessary, replace b with a complement of $a \wedge b$ in $[0, a \vee b]$, so we may assume that $a \wedge b = 0$. However, then $[a] + [b]$ is defined and is equal to $[a \vee b]$. By (GC), there is no loss in generality in assuming that $[a] \preccurlyeq [b]$. However, then $[a] + [b] \preccurlyeq 2[b]$ and hence $[a \vee b] \preccurlyeq 2[b]$. Because $2[b] \subseteq K$, it follows that $a \vee b \in K$. ∎

One can further show that the neutral ideals of M form a chain with K the unique maximal member. However, we shall not make this digression.

COROLLARY 3.4. *M/K satisfies* (IP).

Proof. M/K is simple. ∎

We have yet to relate these results to the situation at hand. We accordingly assume P is a prime ideal of the center of the continuous geometry L, that $M = L/P$, and that J_P is the unique maximal neutral ideal of L such that $P = J_P \cap Z$. We continue to let K be defined as in the preceding text.

COROLLARY 3.5. $J_P = \{a \in L: a/P \in K\}$.

Proof. If $J = \{a \in L: a/P \in K\}$, clearly J is a maximal neutral ideal of L having the property that $J \cap Z = P$. Hence $J = J_P$. ∎

4. A BUNDLE REPRESENTATION

In the reducible case, the dimension function on a continuous geometry is no longer a map from L into the real unit interval; it is a map D from L into the set F of continuous functions from X into $[0,1]$. Let $\underline{0}, \underline{1}$ denote the obvious constant functions in F, by [5, Satz 1.4, p. 112], D satisfies

(1) $\underline{0} \leq D(a) \leq \underline{1}$, $D(0) = \underline{0}$, $D(1) = \underline{1}$.

(2) If $a > 0$, then $D(a) > \underline{0}$.

(3) $D(a \vee b) + D(a \wedge b) = D(a) + D(b)$.

(4) $[a] = [b]$ is equivalent to $D(a) = D(b)$.

(5) $[a] \prec [b]$ is equivalent to $D(a) < D(b)$.

D also preserves up-directed joins [5, pp. 113–115], i.e.,

(6) If $A \subseteq L$ is up-directed, then $D(\bigvee A) = \bigvee_{a \in A} D(a)$ (join taken in F).

There is one final item to note. For J a maximal neutral ideal of L, the lattice L/J is a simple continuous geometry. Hence it has a dimension function D_J in the sense of von Neumann. D is completely determined by the family of dimension functions $\{D_J: J \in Y\}$ and in fact $D_J(a/J) = D(a)(J \cap Z)$ [5, Hilfsatz 3.3, p. 124]. In particular, we shall need the fact that each D_J is a positive modular valuation on L/J, so it induces a metric on L/J.

For elements $a, b \in L$ we will often use the notation $D_J(a)$ to denote $D_J(a/J)$ and $d_J(a,b)$ to denote the associated distance between the elements a/J and b/J. It will be helpful to note that since $d_J(a,b) = D_J(a \vee b) - D_J(a \wedge b)$, it may be expressed as $D_J((a \vee b) \wedge u)$, where u is any complement of $a \wedge b$ in L.

As we remarked earlier, there is a bijection between the set Y of all maximal neutral ideals of L and the set X of all prime ideals of Z given by $J \rightsquigarrow J \cap P$. We shall find it convenient to blur the distinction between Y and X, and we write $J \in X$ when what is really meant is $J \cap Z \in X$. This being said, we make the following definitions. For $a, b \in L$ and any real number ϵ, we let $[D(a) < \epsilon]$ be $\{J \in X: D_J(a) < \epsilon\}$, defining $[d(a,b) < \epsilon]$ similarly. Note that as $[D(a) < \epsilon]$ is equal to the inverse image of $[0, \epsilon)$ under the continuous map $D(a)$, it follows that $[D(a) < \epsilon]$ is open.

Similar remarks show $[D(a) > \epsilon]$ is also open. From the description of $d_J(a,b)$ given in terms of D_J, it follows directly that $[d(a,b) < \epsilon]$ and $[d(a,b) > \epsilon]$ are also open.

In the remainder of this section we shall let T denote the disjoint union of the family of lattices $\{L/J: J \in X\}$. We shall also define for $a \in L$, K a clopen subset of X, and ϵ a real number,

$$B(a, K, \epsilon) = \{b/J: J \in K \text{ and } d_J(a,b) < \epsilon\}.$$

LEMMA 4.1. *The family of sets $B(a, K, \epsilon)$ is a basis for a Hausdorff topology on T. Further, the natural projection π: $T \to X$ is both open and continuous, and the subspace topology on the stalks of T is the usual metric space topology on an irreducible continuous geometry.*

Proof. Suppose that $b/J \in B(a_i, K_i, \epsilon_i)$ for $i = 1, 2$. Set $\beta_i = d_J(a_i, b)$ and note that $\beta_i < \epsilon_i$. Choose a clopen neighborhood M of J which is contained in the open neighborhoods $[d(a_i, b) < (\epsilon_i + \beta_i)/2]$ of J, for each $i = 1, 2$. Because K_1 and K_2 are clopen neighborhoods of J, we may also assume that M was chosen with $M \subseteq K_1 \cap K_2$. We claim that $B(b, M, (\epsilon_i - \beta_i)/2)$ is contained in $B(a_i, K_i, \epsilon_i)$ for $i = 1, 2$. Once this is established, it will follow that for $\delta = \min\{(\epsilon_1 - \beta_1)/2, (\epsilon_2 - \beta_2)/2\}$ the ball $B(b, M, \delta)$ is a basic open neighborhood of b/J contained in the intersection of our original basic open neighborhoods.

To establish our claim, suppose that c/I is in $B(b, M, (\epsilon_1 - \beta_1)/2)$. Then $d_I(b, c) < (\epsilon_1 - \beta_1)/2$. However, because I is an element of M, we have that $d_I(b, a_1) < (\epsilon_1 + \beta_1)/2$, so, by the triangle inequality, we have $d_I(a_1, c) < \epsilon_1$. Therefore as $M \subseteq K_1$ we have that c/I is in $B(a_1, K_1, \epsilon_1)$ as required. We have therefore established that the sets $B(a, K, \epsilon)$ form a basis for a topology.

To see that this topology is Hausdorff let a/J and b/I be any points of T. If $J \neq I$ we can find disjoint clopen subsets of the Stone space, K and M, such that $J \in K$ and $I \in M$. Then $B(a, K, 1)$ and $B(b, M, 1)$ are disjoint open sets separating a/J and b/I. We need only show that a/J and b/J can be separated. Let $d_J(a,b) = \lambda$. Then $[d(a,b) > \lambda/2]$ contains a clopen neighborhood K of J. A simple application of the triangle inequality gives that $B(a, K, \lambda/4)$ and $B(b, K, \lambda/4)$ are disjoint open neighborhoods of a/J and b/J.

That π is open follows because $\pi[B(a, K, \epsilon)] = K$. That π is continuous follows because $\pi^{-1}[K]$ is equal to the union over all $a \in L$ of the basic open sets $B(a, K, 1)$. That the subspace topology on the stalks is the usual metric space topology follows immediately from the definition of the basic open sets $B(a, K, \epsilon)$. ∎

Remark 4.2. While we have defined the topology on T in terms of the dimension function, this was a matter of expediency rather than of necessity—the topology could have been described in purely lattice theoretical terms. The key is in translating the statement $D_J(a) < 1/n$. In the infinite-dimensional case, it turns out that $D_J(a) < 1/n$ is equivalent to the statement

$$\exists b \in L, k \in \mathbb{N} \text{ with } n[(a \vee b)/P] \text{ defined and}$$
$$k[((a \vee b) \wedge u)/P] \text{ not defined,}$$

where $J = J_P$ and u is a complement of a. In the finite-dimensional case $D_J(a) < 1/n$ is equivalent to $(n+1)[a/P]$ being defined.

Recall that a *section* of the bundle (T, X, π) is a map $f\colon X \to T$ such that $\pi \circ f = \mathrm{id}_X$. For each $a \in L$ we have a section $\tilde{a}$ defined by $\tilde{a}(J) = a/J$. Because $\tilde{a}^{-1}[B(b, K, \epsilon)]$ is equal to the intersection of K with $[d(a,b) < \epsilon]$, each $\tilde{a}$ is continuous. We will show that every continuous section is of the form $\tilde{a}$ for some $a \in L$, but we will first need a few technical lemmas.

LEMMA 4.3. *If $a_0 \leq a_1 \leq \cdots$ is an increasing sequence in L with $a = \bigvee a_k$, then for any real number ϵ,*

(1) *If $D_J(a_k) < \epsilon$ for each k and each J, then $D_J(a) \leq \epsilon$ for each J.*

(2) *If $d_J(a_0, a_k) < \epsilon$ for each k and each J, then $d_J(a_0, a) \leq \epsilon$ for each J.*

Proof. (1) As $a_0 \leq a_1 \leq \cdots$ is up-directed, $D(a) = \bigvee_k D(a_k)$, this latter join being taken in F. However, any join in F is pointwise on a dense subset of X (see [5, Anmerkung 1.1, p. 106]. Therefore, $[D(a) \leq \epsilon]$ must contain a dense subset of X. However, $[D(a) \leq \epsilon]$ is closed and so our result follows.

(2) Let u be a complement of a_0 in L. Because L is continuous, the join of the increasing sequence $u \wedge a_0 \leq u \wedge a_1 \leq \cdots$ is $u \wedge \bigvee a_k$ or $u \wedge a$. However, $d_J(a_0, a_k) = D_J(u \wedge a_k)$, so our assumptions give us that $D_J(u \wedge a_k) < \epsilon$ for each k and each J. Applying part (1) we have that $D_J(u \wedge a) \leq \epsilon$ and therefore $d_J(a_0, a) \leq \epsilon$. ∎

LEMMA 4.4. *If f is a continuous section, then for any $\epsilon > 0$ there is an element $a \in L$ such that the distance between a/J and $f(J)$ is strictly less than ϵ for all $J \in X$.*

Proof. Consider the collection $\mathcal{T}$ of all pairs (a, K) such that a is an element of L, K is a clopen subset of X, and the distance between a/J and $f(J)$ is strictly less than ϵ for all points $J \in K$. Define a family

$(a_i, K_1)_I$ of elements of $\mathcal{T}$ to be admissible if the K_i are pairwise disjoint. Let $\mathcal{S}$ be the collection of all admissible families in $\mathcal{T}$. $\mathcal{S}$ is nonempty, since it contains the empty family, and is closed under unions of chains. So by Zorn's lemma there is a maximal admissible family $(a_i, K_i)_I$ is $\mathcal{S}$. We claim that $D = \bigcup K_i$ is a dense open subset of X. Suppose that K is a nonempty clopen subset of X which is disjoint from D, and let J be any point in K. Say $f(J) = a/J$. Whereas f is continuous, $f^{-1}[B(a, K, \epsilon)] \cap K$ will contain a clopen neighborhood M of J. Then (a, M) is a member of $\mathcal{T}$. However, M is disjoint from D, contrary to the maximality of our chosen family.

It does no harm to assume that our maximal family $(a_i, K_i)_I$ was chosen so that a_i/J vanished for all J outside of K_i because we could replace a_i with $a_i \wedge c_i$, where c_i is a central element vanishing outside of K_i and agreeing with 1 on K_i. Let $a = \bigvee a_i$. It is not difficult to see that a/J agrees with a_i/J for each point J in K_i. Therefore the distance between a/J and $f(J)$ is strictly less than ϵ for all $J \in D$. We claim that the distance between a/J and $f(J)$ is less than or equal to ϵ for all points $J \in X$.

Suppose that the distance between a/J and $f(J)$ is $\delta > \epsilon$ and that $f(J) = b/J$. Set

$$A = f^{-1}\big[B(b, X, (\delta - \epsilon)/2)\big]$$

and

$$B = \big[d(a,b) > (\delta + \epsilon)/2\big].$$

Both these sets are open neighborhoods of J, and because D is a dense open set, there is some point I in the intersection of A, B, and D. Then because I is in A, we have the distance between b/I and $f(I)$ is less than $(\delta - \epsilon)/2$, but because I is in D, we have the distance between $f(I)$ and a/I is less than ϵ. So by the triangle inequality, the distance between a/I and b/I must be less than $(\delta + \epsilon)/2$, contradicting that I is in B. ▮

LEMMA 4.5. *Every continuous section of the bundle (T, X, π) is of the form $\tilde{a}$ for some $a \in L$.*

Proof. By the previous lemma, for each natural number n we can find an element x_n of L such that the distance between x_n/J and $f(J)$ is strictly less than $1/2^{n+1}$ for each J in X. An application of the triangle inequality then yields that for any $J \in X$,

$$d_J(x_n, x_{n+1}) < 1/2^{n+1} + 1/2^{n+2} < 1/2^n. \qquad (4.1)$$

For natural numbers n, r define

$$y_{n,r} = x_n \vee \cdots \vee x_{n+r},$$
$$z_{n,r} = x_n \wedge x_{n+1} \cdots \wedge x_{n+r},$$
$$y_n = \bigvee_r y_{n,r} \quad \text{and} \quad z_n = \bigwedge_r z_{n,r}.$$

Using (4.1) and the fact that for any a, b, c we have $d_J(a \vee b, a \vee c) \leq d_J(b, c)$ [5, Satz 6.2, p. 46], we have

$$\begin{aligned} d_J(y_{n,r}, y_{n,r+1}) &= d_J(y_{n,r} \vee x_{n+r}, y_{n,r} \vee x_{n+r+1}) \\ &\leq d_J(x_{n+r}, x_{n+r+1}) \\ &< 1/2^{n+r}. \end{aligned} \tag{4.2}$$

Next, using (4.2), we have

$$\begin{aligned} d_J(y_{n,r}, y_{n,r+k}) &\leq d_J(y_{n,r}, y_{n,r+1}) + \cdots + d_J(y_{n,r+k-1}, y_{n,r+k}) \\ &< 1/2^{n+r} + \cdots + 1/2^{n+r+k-1} \\ &< 1/2^{n+r-1}. \end{aligned}$$

We may therefore apply Lemma 4.3 to the sequence $y_{n,r} \leq y_{n,r+1} \leq \cdots$ to get

$$d_J(y_{n,r}, y_n) \leq 1/2^{n+r-1}.$$

In particular, for $r = 0$ we get

$$d_J(x_n, y_n) \leq 1/2^{n-1},$$

and dually,

$$d_J(x_n, z_n) \leq 1/2^{n-1}.$$

Therefore by the triangle inequality

$$d_J(y_n, z_n) \leq 1/2^{n-2}. \tag{4.3}$$

Define

$$y = \bigwedge y_n \quad \text{and} \quad z = \bigvee z_n.$$

For any natural numbers m, n, set $k = \max\{m, n\}$. From the definition, it is easy to see that $z_n \leq x_k$ and $x_k \leq y_m$. Because this is true for all m, n we have $z_n \leq y \leq y_n$. However, it is obvious from the definition that x_n also lies between z_n and y_n. So by (4.3),

$$d_J(y, x_n) \leq d(z_n, y_n) \leq 1/2^{n-2}. \tag{4.4}$$

Let $d_J(y, f)$ denote the distance between y/J and $f(J)$. Because x_n was chosen so that $d_J(x_n, f) < 1/2^{n+1}$, we have by the triangle inequality and (4.4),

$$\begin{aligned} d_J(y, f) &\leq d_J(y, x_n) + d_J(x_n, f) \\ &\leq 1/2^{n-2} + 1/2^{n+1} \\ &< 1/2^{n-3}. \end{aligned}$$

Because this holds for all natural numbers n, we have that $d_J(y, f) = 0$. Thus $f(J) = \tilde{y}(J)$ for all points J, and therefore $\tilde{y} = f$. ∎

THEOREM 4.6. *L is isomorphic to the continuous sections of the bundle (T, X, π).*

Proof. We have seen that for each $a \in L$ the section $\tilde{a}$ is continuous and that all continuous sections arise in this manner. We need only show for $a \neq b$ that $\tilde{a} \neq \tilde{b}$. However, this follows from Maeda's result that the intersection of $\{J: J \in X\}$ is trivial. ∎

EXAMPLE 4.7. In sharp contrast to the Pierce sheaf, equalizers in our bundle are not usually clopen. Consider the product $L = \Pi L_n$ of n-dimensional geometries that was discussed in Example 2.1. Let $a \in L$ be such that a_n is an atom in L_n for each n. Then $n[a/P]$ is defined for all n if and only if P is a nonprincipal prime ideal. Thus $\tilde{a}$ is 0 only at those J corresponding to nonprincipal prime ideals.

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