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Mathematical Logic

The modal logic of $\beta(\mathbb{N})$

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Abstract Let $\beta(\mathbb{N})$ denote the Stone–Čech compactification of the set $\mathbb{N}$ of natural numbers (with the discrete topology), and let $\mathbb{N}^*$ denote the remainder $\beta(\mathbb{N}) - \mathbb{N}$. We show that, interpreting modal diamond as the closure in a topological space, the modal logic of $\mathbb{N}^*$ is **S4** and that the modal logic of $\beta(\mathbb{N})$ is **S4.1.2**.

Keywords Modal logic · Topology · Stone–Čech compactification

Mathematics Subject Classification (2000) 03B45 · 03G05 · 54D35

1 Introduction

It was shown by McKinsey and Tarski [12,13] that if we interpret modal diamond as the closure in a topological space, then the modal logic of topological spaces is Lewis' well-known modal system **S4**. Their classic 1944 result states that **S4** is in fact the modal logic of any dense-in-itself metrizable space. In particular, **S4** is the modal logic of the Cantor space $\mathcal{C}$, the real line $\mathbb{R}$, and the rational line $\mathbb{Q}$. A modern proof of completeness of **S4** with respect to $\mathcal{C}$ is given in [1,14], that with respect to $\mathbb{R}$ in [1,4], and that with respect to $\mathbb{Q}$ in [3]. On the other hand, completeness issues with respect to important non-metrizable spaces have not been raised so far in the literature. In this

To the memory of Lazo Zambakhidze (1942–2008).

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note we concentrate on an important non-metrizable space $\beta(\mathbb{N})$—the Stone–Čech compactification of the set $\mathbb{N}$ of natural numbers (with the discrete topology).

Our main result states that under the set-theoretic assumption that each infinite maximal almost disjoint family of subsets of $\mathbb{N}$ has cardinality 2^ω, the modal logic of the remainder $\mathbb{N}^* = \beta(\mathbb{N}) - \mathbb{N}$ is **S4**. From this, it follows that under the same assumption, the modal logic of $\beta(\mathbb{N})$ is **S4.1.2**, which is obtained by adding to **S4** the axiom $\Box\Diamond p \leftrightarrow \Diamond\Box p$. The set theoretic assumption that each infinite maximal almost disjoint family of subsets of $\mathbb{N}$ has cardinality 2^ω is not provable in ZFC (the Zermelo–Fraenkel Set Theory with the Axiom of Choice). However, this assumption is known to be a consequence of Martin's Axiom [11, p. 57], and it is a simple consequence of the Continuum Hypothesis. It is an open problem whether our main result holds true within ZFC.

The paper is organized as follows. In Sect. 2 we recall basic facts about relational semantics of **S4**, including completeness of **S4** with respect to finite quasi-trees. In Sect. 3 we recall basics about the Boolean algebra $\wp(\mathbb{N})/_{\mathrm{Fin}}$ of the powerset of $\mathbb{N}$ modulo the ideal of finite subsets of $\mathbb{N}$. Section 4 is the heart of the paper in which we show that there exists an interior map from the Stone space of $\wp(\mathbb{N})/_{\mathrm{Fin}}$ onto each finite quasi-tree. Since $\mathbb{N}^*$ is homeomorphic to the Stone space of $\wp(\mathbb{N})/_{\mathrm{Fin}}$, as a corollary we obtain that **S4** is the modal logic of $\mathbb{N}^*$, thus adding to completeness results of McKinsey and Tarski and others. In Sect. 5 we show how to adjust the proof of Sect. 4 to obtain that **S4.1.2** is the modal logic of $\beta(\mathbb{N})$. We conclude the paper by mentioning several consequences of our results.

2 Preliminaries

We recall that **S4** is the least set of formulas containing the Boolean tautologies, the axioms:

$\Box(p \to q) \to (\Box p \to \Box q)$
$\Box p \to p$
$\Box\Box p \to \Box p$

and closed under Modus Ponens ($\varphi, \varphi \to \psi/\psi$) and Necessitation ($\varphi/\Box\varphi$). Relational frames of **S4** are quasi-ordered sets $\langle X, \leq\rangle$; that is, X is a nonempty set and $\leq$ is reflexive and transitive. A quasi-ordered set $\langle X, \leq\rangle$ is called *rooted* if there exists $r \in X$—called a *root* of X—such that $r \leq x$ for each $x \in X$. For $\langle X, \leq\rangle$ a quasi-ordered set, $x \in X$, and $A \subseteq X$, let $\downarrow x = \{y \in X : y \leq x\}$ and let $\downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\}$. We call $A \subseteq X$ a *chain* if $x \leq y$ or $y \leq x$ for all $x, y \in A$. A quasi-ordered set $\langle X, \leq\rangle$ is called a *tree* if $\langle X, \leq\rangle$ is a rooted partially ordered set and $\downarrow x$ is a chain for each $x \in X$. For $x \in X$, let $C[x] = \{y \in X : x \leq y \text{ and } y \leq x\}$. We call $C \subseteq X$ a *cluster* if $C = C[x]$ for some $x \in X$. Define an equivalence relation $\sim$ on X by $x \sim y$ if $C[x] = C[y]$. Let $X/{\sim}$ denote the quotient of X under $\sim$—called the *skeleton* of X. We call X a quasi-tree if $X/{\sim}$ is a tree. A well-known result in modal logic states that **S4** is complete with respect to finite quasi-trees (see, e.g., [4, Cor. 6]).

For $\langle X, \leq\rangle$ a quasi-ordered set, a subset U of X is called an *upset* of X if $x \in U$ and $x \leq y$ imply $y \in U$. It is well-known that every quasi-ordered set $\langle X, \leq\rangle$ can be viewed as a topological space by letting the upsets of X be open subsets of X. In fact, quasi-ordered sets are very special topological spaces—called *Alexandroff spaces*—in which the intersection of any family of opens is again open. Thus, relational semantics for **S4** can be viewed as a special case of topological semantics for **S4**.

For two topological spaces X and Y, we recall that a map $f : X \to Y$ is *continuous* if for each open V of Y, we have $f^{-1}(V)$ is open in X; that f is *open* if U open in X implies $f(U)$ is open in Y; and that f is *interior* if it is both continuous and open. It is well-known (see, e.g., [8, Thm. 2.1.8]; [2, Prop. 2.9]) that onto interior maps preserve validity of modal formulas. This fact is very useful in proving topological completeness results. Indeed, since **S4** is complete with respect to finite quasi-trees, in order to obtain the McKinsey–Tarski result that **S4** is complete with respect to any dense-in-itself metrizable space X, it is sufficient to construct an interior map from X onto every finite quasi-tree. Similarly, in order to prove completeness of **S4** with respect to the remainder $\mathbb{N}^*$ of the Stone–Čech compactification $\beta(\mathbb{N})$ of $\mathbb{N}$, it is sufficient to construct an interior map from $\mathbb{N}^*$ onto every finite quasi-tree. Then, by completeness of **S4** with respect to finite quasi-trees, if a formula φ is not provable in **S4**, there exists a finite quasi-tree $\langle X, \leq\rangle$ refuting φ. Viewing $\langle X, \leq\rangle$ as a topological space, there is an interior onto map $f : \mathbb{N}^* \to X$. And since interior onto maps preserve validity of formulas and φ is refuted on X, it can also be refuted on $\mathbb{N}^*$. This is exactly what our strategy is going to be: Assuming that each infinite maximal almost disjoint family of subsets of $\mathbb{N}$ has cardinality 2^ω, we will build an interior map from $\mathbb{N}^*$ onto every finite quasi-tree X; the completeness of **S4** with respect to $\mathbb{N}^*$ will follow immediately. We then show how to use this result to obtain completeness of **S4.1.2** with respect to $\beta(\mathbb{N})$.

3 $\mathbb{N}^*$ and $\wp(\mathbb{N})/_{\mathrm{Fin}}$

It is well-known (see, e.g., [7, pp. 230–232]; [10, p. 95]) that $\beta(\mathbb{N})$ can be thought of as the Stone space of the Boolean algebra $\wp(\mathbb{N})$ of subsets of $\mathbb{N}$. Since $\mathbb{N}^*$ is a closed subset of $\beta(\mathbb{N})$, by the Stone duality, it is the Stone space of a quotient algebra of $\wp(\mathbb{N})$. In fact, $\mathbb{N}^*$ is the Stone space of the Boolean algebra $\wp(\mathbb{N})/_{\mathrm{Fin}}$, where Fin is the ideal of finite subsets of $\mathbb{N}$ (see [10, p. 95]). In this section we consider basic properties of $\wp(\mathbb{N})/_{\mathrm{Fin}}$, which will be needed to prove completeness of **S4** with respect to $\mathbb{N}^*$. For more detail see [10, pp. 78–82].

Proposition 3.1 *$\wp(\mathbb{N})/_{\mathrm{Fin}}$ is homogeneous, which means that for each nonzero $b \in \wp(\mathbb{N})/_{\mathrm{Fin}}$, the interval $[0, b]$ is isomorphic as a Boolean algebra to $\wp(\mathbb{N})/_{\mathrm{Fin}}$.*

Proof As $b \neq 0$ we have $b = [A]$ for some infinite subset $A \subseteq \mathbb{N}$. Then there is a bijection $\varphi : A \to \mathbb{N}$. The map sending $[S]$ to $[\varphi^{-1}(S)]$, for each $S \subseteq \mathbb{N}$, is the required isomorphism from $\wp(\mathbb{N})/_{\mathrm{Fin}}$ to $[0, b]$. □

We recall that a set P of nonzero elements of a Boolean algebra B is *orthogonal* if any two distinct elements of P meet to zero, that P is a *partition* of $b \in B$ if P is an orthogonal set whose join is b, and that P is a *partition of unity* if it is a partition of the top element 1 of B. It is well known that Zorn's lemma implies every orthogonal set can be extended to a maximal orthogonal set, and that maximal orthogonal sets are exactly the partitions of unity.

A Boolean algebra B is said to satisfy the *countable separation property* [10, p. 79] if for any countable subsets D, E of B with $d \wedge e = 0$ for each $d \in D$ and $e \in E$, there is an element $b \in B$ with $d \leq b$ for each $d \in D$ and $e \leq -b$ for each $e \in E$.

Proposition 3.2 *If a Boolean algebra B satisfies the countable separation property and P is an infinite orthogonal set of B, then the ideal I generated by P is not a maximal ideal.*

Proof As P is infinite we can find two disjoint countable subsets D, E of P. As B satisfies the countable separation property, there is some $b \in B$ with $d \leq b$ for each $d \in D$ and $e \leq -b$ for each $e \in E$. As there are infinitely many members of the orthogonal set P lying beneath b, it cannot be the case that b lies beneath the join of finitely many members of P. So b does not belong to I. Similarly, $-b \notin I$. Thus I is not maximal. □

Our primary concern will be with orthogonal sets that are a partition of some $b \neq 0$ in $\wp(\mathbb{N})/_{\mathrm{Fin}}$. Our first facts below are obtained using only the axioms of ZFC. They are proved in the case when $b = 1$ in [10, p. 78], and the generalization to any $b \neq 0$ is a direct consequence of the homogeneity of $\wp(\mathbb{N})/_{\mathrm{Fin}}$.

Proposition 3.3 *If b is a nonzero element of $\wp(\mathbb{N})/_{\mathrm{Fin}}$, then there is a partition of b of cardinality 2^ω and each infinite partition of b is uncountable.*

As $\wp(\mathbb{N})/_{\mathrm{Fin}}$ itself has cardinality 2^ω, the above result says that if κ is the cardinality of an infinite partition of b, then $\omega_1 \leq \kappa \leq 2^\omega$, and that this upper bound 2^ω is realized by at least one partition of b. It is, however, consistent with ZFC that a partition of b can have cardinality strictly less than 2^ω. In our argument we require that each infinite partition of b has cardinality 2^ω. This is equivalent to the well-studied assumption in infinitary combinatorics that each infinite maximal almost disjoint family of subsets of $\mathbb{N}$ has cardinality 2^ω. We refer to this set-theoretic assumption as $(\mathfrak{a} = 2^\omega)$ as it is common to denote the least cardinality of an infinite maximal almost disjoint family of subsets of $\mathbb{N}$ by $\mathfrak{a}$. It is well-known that $(\mathfrak{a} = 2^\omega)$ follows from the Continuum Hypothesis (CH) or Martin's Axiom (MA), which is weaker than (CH). However, $(\mathfrak{a} = 2^\omega)$ is not provable in ZFC. That $(\mathfrak{a} = 2^\omega)$ arises is not surprising. It is standard to consider the behavior of $\mathbb{N}^*$ under further set theoretic assumptions [17].

4 The modal logic of $\mathbb{N}^*$

In this section we prove our main result that, under $(\mathfrak{a} = 2^\omega)$, for each finite quasi-tree Q, there exists an interior map from $\mathbb{N}^*$ onto Q. As a corollary, we obtain that **S4** is the modal logic of $\mathbb{N}^*$.

For an integer m, let $\{1, \ldots, m\}^*$ be all finite sequences σ of $1, \ldots, m$. We call the number of terms in the sequence σ its *length*. The unique sequence with no terms is called the *empty sequence* and denoted Λ.

Let T be a finite tree. We call T *regular* if the branching size of each node is the same. Given integers $m, n \geq 1$ let $T_{m,n}$ denote the regular tree of branching size m and depth $n + 1$. We can think of the nodes of this tree as all σ where σ belongs to $\{1, \dots, m\}^*$ and has length at most n. The root is the node Λ, and the m children of the node σ are the nodes $\sigma 1, \dots, \sigma m$.

Let Q be a quasi-tree. We call Q *regular* if $Q/{\sim}$ is a regular tree. Given integers $m, n, k \geq 1$, let $Q_{m,n,k}$ be the regular quasi-tree of branching size m, depth $n+1$, and cluster size k obtained by replacing each node σ of the tree $T_{m,n}$ by a cluster of size k. A key fact, established in [4, Lem. 5], is that for each finite quasi-tree Q, there are m, n, k such that Q is an interior image of $Q_{m,n,k}$. So to show each finite quasi-tree is an interior image of $\mathbb{N}^*$, it is enough to show each $Q_{m,n,k}$ is such an interior image.

It is our goal to show that given integers $m, n, k \geq 1$, there exists an interior onto map $f : \mathbb{N}^* \to Q_{m,n,k}$. The proof consists of several stages. To begin, take an arbitrary, but fixed, branching size $m \geq 1$. We first build an infinite sequence of partitions of unity of $\wp(\mathbb{N})/_{\mathrm{Fin}}$ having a number of specific technical properties. This sequence is used to build a tree of ideals of $\wp(\mathbb{N})/_{\mathrm{Fin}}$ with branching size m and infinite depth. This tree of ideals is used to construct an interior map f from the Stone space of $\wp(\mathbb{N})/_{\mathrm{Fin}}$ onto any tree $T_{m,n}$. Finally we show this map can be modified to provide the required interior map from the Stone space of $\wp(\mathbb{N})/_{\mathrm{Fin}}$ onto any quasi-tree $Q_{m,n,k}$. We begin with a definition to describe the technical properties required of our partitions of unity.

Definition 4.1 Suppose $b \in \wp(\mathbb{N})/_{\mathrm{Fin}}$ and P is a partition of b. For each $c \in \wp(\mathbb{N})/_{\mathrm{Fin}}$ set

$$\begin{aligned} \mathrm{Support}_P(c) &= \{p \in P : c \wedge p \neq 0\}. \\ \mathrm{Infinite}(P) &= \{c : c \leq b \text{ and } \mathrm{Support}_P(c) \text{ is infinite}\}. \end{aligned}$$

Note that if P is a partition of b, then the ideal generated by P consists exactly of those elements of the interval $[0, b]$ whose support in P is finite, and the remaining elements of $[0, b]$ are in $\mathrm{Infinite}(P)$. The following is the key technical result where we require $(\mathfrak{a} = 2^\omega)$ to control the size of partitions of an element b.

Lemma 4.2 *Assume* $(\mathfrak{a} = 2^\omega)$. *For P an infinite partition of $b \in \wp(\mathbb{N})/_{\mathrm{Fin}}$ and a natural number m, there are sets $P_1, \dots, P_m$ and maps $f_1, \dots, f_m$ with*

(1) $P_1 \cup \dots \cup P_m = P$ *and* $P_i \cap P_j = \emptyset$ *for each* $i \neq j$.
(2) $f_i : \mathrm{Infinite}(P) \to P_i$ *is a 1-1 map for each* $i \leq m$.
(3) $f_i(c) \in \mathrm{Support}_P(c)$ *for each* $c \in \mathrm{Infinite}(P)$ *and each* $i \leq m$.

We call $P_1, \dots, P_m$ and $f_1, \dots, f_m$ a supportive family *for P.*

Proof It is sufficient to find maps $f_i : \mathrm{Infinite}(P) \to P$ for $i \leq m$ such that each f_i is 1-1, the images of the f_i are pairwise disjoint, and $f_i(c) \in \mathrm{Support}_P(c)$ for each $c \in \mathrm{Infinite}(P)$ and $i \leq m$. The required sets $P_1, \dots, P_m$ are then produced by extending the disjoint images of these functions to a pairwise disjoint covering of P.

Suppose $\mathrm{Infinite}(P)$ has cardinality κ and c_λ $(\lambda \in \kappa)$ enumerates this set. We define $f_1(c_\beta), \dots, f_m(c_\beta)$ by transfinite recursion on $\beta < \kappa$ assuming $f_1(c_\lambda), \dots, f_m(c_\lambda)$ are defined for all $\lambda < \beta$.

Let $\beta < \kappa$. As $c_\beta \in \text{Infinite}(P)$, using infinite distributivity, $\{c_\beta \wedge p : p \in \text{Support}_P(c_\beta)\}$ is an infinite partition of c_β. By the assumption ($\mathfrak{a} = 2^\omega$), this partition has cardinality 2^ω, hence $\text{Support}_P(c_\beta)$ has cardinality 2^ω. But $\beta < \kappa \leq 2^\omega$, so $\{f_i(c_\lambda) : i \leq m,\ \lambda < \beta\}$ has cardinality strictly less than 2^ω. So there are elements $p_{\beta 1}, \ldots, p_{\beta m}$ belonging to $\text{Support}_P(c_\beta)$ and not in $\{f_i(c_\lambda) : i \leq m,\ \lambda < \beta\}$. Set $f_i(c_\beta) = p_{\beta i}$. □

Lemma 4.3 *There is an infinite sequence of partitions of unity $P_0, P_1, \ldots$ such that $P_0 = \{1\}$ and for each $b \in P_n$*

(1) *$P^b = \downarrow b \cap P_{n+1}$ is an infinite partition of b.*
(2) *There are $P_1^b, \ldots, P_m^b$ and $f_1^b, \ldots, f_m^b$ supportive for P^b.*
(3) *$c \wedge f_j^b(c)$ has infinite support in P_{n+2} for each $j \leq m$ and $c \in$* Infinite(P^b).

Proof We define this sequence of partitions of unity, and the associated supportive families, by recursion. Let $P_0 = \{1\}$ and let P_1 be any infinite partition of unity. Then Lemma 4.2 supplies supportive $P_1^1, \ldots, P_m^1$ and $f_1^1, \ldots, f_m^1$.

Suppose we have defined partitions of unity $P_0, \ldots, P_n$ and for each b belonging to some P_i with $i \leq n-1$ we have $P^b = \downarrow b \cap P_{i+1}$ is an infinite partition of b. Suppose also that if b belongs to P_i for some $i \leq n-1$, we have supportive $P_1^b, \ldots, P_m^b$ and $f_1^b, \ldots, f_m^b$ for P^b and if $i \leq n-2$, condition 3 holds for these maps.

We will define a partition of unity P_{n+1}. This must be done so that for each $b \in P_n$, we have $P^b = \downarrow b \cap P_{n+1}$ is an infinite partition of b. When defining P_{n+1} we must also make sure for each $d \in P_{n-1}$ and each $c \leq d$ of infinite support in P^d, that $c \wedge f_j^d(c)$ has infinite support in P_{n+1} for each $j \leq m$. Finally, for each $b \in P_n$ we must create a supportive family $P_1^b, \ldots, P_m^b$ and $f_1^b, \ldots, f_m^b$ for P^b.

Suppose $b \in P_n$. We claim there is at most one $d \in P_{n-1}$, one $c \leq d$ of infinite support in P^d, and one $j \leq m$ with $b = f_j^d(c)$. Indeed, for such d, c, j as $b = f_j^d(c)$ we must have $b \in P^d$. Since the elements of P_{n-1} are pairwise disjoint, this d must be the unique element of P_{n-1} lying above b. As the images of the $f_1^d, \ldots, f_m^d$ are pairwise disjoint, there can be at most one $j \leq m$ with b in the image of f_j^d. Then because f_j^d is 1-1, there is at most one c with $b = f_j^d(c)$.

Suppose $b \in P_n$ and there are d, c, j as above with $b = f_j^d(c)$. Then as $f_j^d(c)$ belongs to the support of c in P^d, we have $b \wedge c \neq 0$. By Proposition 3.3 there is an infinite partition of $b \wedge c$. Extend this to a maximal orthogonal set in the interval $[0, b]$, hence to a partition P^b of b. Note that the support of $b \wedge c$ in P^b is infinite. If $b \in P_n$ and there are no such d, c, j, let P^b be any infinite partition of b.

Let $P_{n+1} = \bigcup\{P^b : b \in P_n\}$. Each P^b is an orthogonal set, and elements from different sets P^b are also orthogonal, so P_{n+1} is orthogonal. As the join of P^b is b, it follows that the join of P_{n+1} equals that of P_n, hence is 1. So P_{n+1} is a partition of unity. Also, for each $b \in P_n$ we have by construction that $\downarrow b \cap P_{n+1}$ equals P^b, hence is an infinite partition of b. Suppose $d \in P_{n-1}$, $c \leq d$ has infinite support in P^d, and $j \leq m$. Then for $b = f_j^d(c)$, we have constructed P^b so that $b \wedge c$ has infinite support in P^b, hence this element has infinite support in P_{n+1}. For each $b \in P_n$, it remains only to construct a supportive family $P_1^b, \ldots, P_m^b$ and $f_1^b, \ldots, f_m^b$ for P^b. But this follows directly from Lemma 4.2. □

We use this setup to build a tree of ideals of $\wp(\mathbb{N})/_{\mathrm{Fin}}$.

Definition 4.4 For each $\sigma \in \{1, \dots, m\}^*$ define S_σ by setting

$$S_\Lambda = \{1\},$$
$$S_{\sigma j} = \bigcup\{P_j^b : b \in S_\sigma\}.$$

Here, σj is the string formed by concatenating j to the end of the string σ. Having defined S_σ for each σ we let I_σ be the ideal of $\wp(\mathbb{N})/_{\mathrm{Fin}}$ generated by S_σ.

Lemma 4.5 *For the ideals I_σ constructed above*

(1) $I_\sigma \subseteq I_\rho$ *if* σ *extends* ρ.
(2) $I_\sigma \cap I_\rho = \{0\}$ *unless one of* σ, ρ *extends the other.*
(3) $1 \in I_\Lambda - \bigvee_{j=1}^m I_j$.
(4) $a \in I_\sigma - \bigvee_{j=1}^m I_{\sigma j} \Rightarrow$ *for each* $i \leq m$ *there exists* $d \leq a$ *with* $d \in I_{\sigma i} - \bigvee_{j=1}^m I_{\sigma i j}$.

Proof For the first condition, it is enough to show $I_{\sigma j} \subseteq I_\sigma$ for any σ and any $j \leq m$. But if $b \in S_\sigma$, then P_j^b is contained in $\downarrow b$. So each generator of $I_{\sigma j}$ lies beneath a generator of I_σ, hence $I_{\sigma j} \subseteq I_\sigma$. For the second condition, it is enough to show $I_{\sigma i} \cap I_{\sigma j} = \{0\}$ for any σ and any $i \neq j \leq m$. Suppose $b, c \in S_\sigma$, and $p \in P_i^b$, $q \in P_j^c$. If $b \neq c$ then as $p \leq b, q \leq c$ and b, c are orthogonal, p, q are orthogonal. If $b = c$ then P_i^b and P_j^b are distinct, hence disjoint subsets of P^b, so p, q are orthogonal. Thus every element in the generating set of $I_{\sigma i}$ is orthogonal to every element in the generating set of $I_{\sigma j}$, and it follows that $I_{\sigma i} \cap I_{\sigma j} = \{0\}$. For the third condition, 1 belongs to the generating set S_Λ of I_Λ and as the generating set P_1 of $\bigvee_{j=1}^m I_j$ is an infinite partition of unity, 1 does not belong to this join.

For the final condition, suppose σ has length n. As $a \in I_\sigma$ we have $a \leq b_1 \vee \cdots \vee b_k$ for some $b_1, \dots, b_k \in S_\sigma$. Hence $a = (a \wedge b_1) \vee \cdots \vee (a \wedge b_k)$. Since a does not belong to $\bigvee_{j=1}^m I_{\sigma j}$, there is some $b \in S_\sigma$ with $a \wedge b$ not belonging to $\bigvee_{j=1}^m I_{\sigma j}$. As $b \in S_\sigma$ and $P^b = \bigcup_{j=1}^m P_j^b$ we have $P^b \subseteq \bigcup_{j=1}^m S_{\sigma j}$ hence P^b is contained in $\bigvee_{j=1}^m I_{\sigma j}$. As $a \wedge b$ does not belong to $\bigvee_{j=1}^m I_{\sigma j}$ and clearly lies under b, the support of $a \wedge b$ in P^b must be infinite. Let $c = a \wedge b$. Condition 3 of Lemma 4.3 gives that $d = c \wedge f_i^b(c)$ has infinite support in P_{n+2}. As $f_i^b(c)$ belongs to the image of f_i^b, it belongs to P_i^b, and as $b \in S_\sigma$, we have $f_i^b(c)$ belongs to $S_{\sigma i}$, and hence also to the ideal $I_{\sigma i}$ it generates. As $d \leq f_i^b(c)$ we have $d \in I_{\sigma i}$. Since the support of d in P_{n+2} is infinite and $\bigvee_{j=1}^m I_{\sigma i j}$ is generated by a subset of P_{n+2}, it follows that d does not belong to this join. □

Let X be the Stone space of ultrafilters of the Boolean algebra $\wp(\mathbb{N})/_{\mathrm{Fin}}$. We recall that $\{\phi(a) : a \in \wp(\mathbb{N})/_{\mathrm{Fin}}\}$ forms a basis of clopen (simultaneously closed and open) subsets for the topology on X, where $\phi(a) = \{x \in X : a \in x\}$. For an ideal I of $\wp(\mathbb{N})/_{\mathrm{Fin}}$, let $U_I = \bigcup\{\phi(a) : a \in I\}$ denote the open subset of X associated with I by the Stone duality.

Definition 4.6 For $x \in X$ and $n \geq 1$ define

(1) $\Sigma(x) = \{\sigma : x \in U_{I_\sigma}\}$.
(2) $\Sigma_n(x) = \{\sigma : x \in U_{I_\sigma}$ and σ has length at most $n\}$.

Lemma 4.7 *For $x \in X$ and $n \geq 1$*

(1) $\Lambda \in \Sigma(x)$.
(2) *If $\sigma, \rho \in \Sigma(x)$ then one of σ, ρ is an extension of the other.*
(3) *$\Sigma_n(x)$ has a unique element of maximal length.*

We let $\sigma_n(x)$ be the unique element of maximal length in $\Sigma_n(x)$.

Proof The first statement follows as $1 \in S_\Lambda$, so I_Λ is all of $\wp(\mathbb{N})/_{\mathrm{Fin}}$. For the second, if neither σ, ρ extends the other, then by Lemma 4.5 we have $I_\sigma \cap I_\rho = \{0\}$, and this gives $U_{I_\sigma} \cap U_{I_\rho} = \emptyset$. For the third, $\Sigma_n(x)$ trivially must have elements of maximal length. That there is only one element of maximal length follows from the second condition. □

Proposition 4.8 *For $n \geq 1$, the map $f : X \to T_{m,n}$ defined by $f(x) = \sigma_n(x)$ is interior and onto.*

Proof This map is well defined. To see it is continuous, since principal upsets of $T_{m,n}$ form a basis for the topology on $T_{m,n}$, it is enough to show the inverse image of a principal upset is open. For any σ of length at most n, the principal upset $\uparrow\sigma$ in the tree $T_{m,n}$ consists of all ρ where ρ is an extension of σ with length at most n. Thus $f^{-1}(\uparrow\sigma)$ is all $x \in X$ with $\sigma_n(x)$ an extension of σ. This is exactly those x belonging to U_{I_σ}. Thus $f^{-1}(\uparrow\sigma) = U_{I_\sigma}$ so f is continuous.

To see f is open, it is enough to show that for each $a \in \wp(\mathbb{N})/_{\mathrm{Fin}}$, the image of the basic open set $\phi(a)$ under f is an upset of $T_{m,n}$. To establish this, it is enough to show that if σ has length at most $n-1$ and $\sigma \in f[\phi(a)]$, then for each $i \leq m$ we have $\sigma i \in f[\phi(a)]$. As $\sigma \in f[\phi(a)]$, there is $x \in \phi(a)$ with $f(x) = \sigma$. This means $\sigma_n(x) = \sigma$, so $x \in U_{I_\sigma} - \bigcup_{j=1}^m U_{I_{\sigma j}}$. As U_{I_σ} is open, there is a basic open $\phi(e)$ with $x \in \phi(e)$ and $\phi(e) \subseteq U_{I_\sigma}$. This implies $e \in I_\sigma$. As $x \in \phi(e)$ and $x \notin \bigcup_{j=1}^m U_{I_{\sigma j}}$, we also have $e \notin \bigvee_{j=1}^m I_{\sigma j}$. Then by condition 4 of Lemma 4.5 there is $d \leq e$ with $d \in I_{\sigma i} - \bigvee_{j=1}^m I_{\sigma i j}$. Then $\phi(d) \subseteq U_{I_{\sigma i}}$ and $\phi(d) \not\subseteq \bigcup_{j=1}^m U_{I_{\sigma i j}}$. Let $y \in \phi(d)$ with $y \notin \bigcup_{j=1}^m U_{I_{\sigma i j}}$. Then $y \in \phi(a)$ and $f(y) = \sigma i$.

It is left to be shown that f is onto. Since f is open and the whole of $T_{m,n}$ is the only open set containing the root Λ, it is sufficient to show there is some $x \in X$ with $f(x) = \Lambda$. But condition 3 of Lemma 4.5 says $\bigcup_{j=1}^m U_{I_j}$ is not equal to all of X, and this provides the result. □

Lemma 4.9 *For any σ, $U_{I_\sigma} - \bigcup_{j=1}^m U_{I_{\sigma j}}$ has no isolated points in the subspace topology.*

Proof Suppose the set $Y = U_{I_\sigma} - \bigcup_{j=1}^m U_{I_{\sigma j}}$ has an isolated point x. This means there is some open subset of X that intersects Y only in the point x. As x belongs to the open set U_{I_σ} we may choose this open set to be a basic open set contained in U_{I_σ}, hence of the form $\phi(a)$ for some $a \in I_\sigma$. As $a \in I_\sigma$ we have $\{e \in S_\sigma : a \wedge e \neq 0\}$ is finite, and $a = \bigvee\{a \wedge e : e \in S_\sigma\}$. As we have expressed a as a finite join, this translates into expressing $\phi(a)$ as a finite union. As $x \in \phi(a)$, this means x belongs to one of the sets in this union. So there is some $b \in S_\sigma$ with $x \in \phi(a \wedge b)$. As $x \notin \bigcup_{j=1}^m U_{I_{\sigma j}}$ we have $a \wedge b \notin \bigvee_{j=1}^m I_{\sigma j}$. Since $\bigvee_{j=1}^m I_{\sigma j}$ contains P^b, $a \wedge b$ has infinite support in P^b.

Let $c = a \wedge b$ and $Q = \{c \wedge h : h \in \text{Support}_{P^b}(c)\}$. As P^b is a partition of b we have Q is a partition of c, and as c has infinite support in P^b, by definition Q is infinite. As the interval $[0, c]$ is isomorphic to $\wp(\mathbb{N})/_{\text{Fin}}$, by Proposition 3.2, the ideal generated by Q is not a maximal ideal of this interval. So there are distinct ultrafilters y, z of this interval with both y, z disjoint from Q. Extend y, z to ultrafilters y', z' of $\wp(\mathbb{N})/_{\text{Fin}}$. As $y' \cap \downarrow c = y$ and $z' \cap \downarrow c = z$ we have y', z' are distinct. As $c \in y', z'$ we have $y', z' \in \phi(c)$, hence $y', z' \in \phi(a)$. We claim $y', z' \notin \bigcup_{j=1}^m U_{I_{\sigma j}}$. We show this only for y', that it is true also of z' follows by symmetry.

If $y' \in \bigcup_{j=1}^m U_{I_{\sigma j}}$, then there is some element of $\bigvee_{j=1}^m I_{\sigma j}$ belonging to y'. As $\bigvee_{j=1}^m I_{\sigma j}$ is generated by $S = \bigcup\{P^d : d \in S_\sigma\}$ some finite join of elements of this generating set belongs to y', and since y' is a maximal, hence prime, filter we have that some member h of this generating set S belongs to y'. As $c, h \in y'$ we have $c \wedge h \in y'$, hence $c \wedge h \in y' \cap \downarrow c = y$. In particular $c \wedge h \neq 0$. Because $h \in S$ we have $h \in P^d$ for some $d \in S_\sigma$, and as $0 \neq c \wedge h \leq b \wedge h$ it must be that $h \in P^b$ since the elements of S_σ are orthogonal. Then as $c \wedge h \neq 0$ we have h belongs to $\text{Support}_{P^b}(c)$. Thus $c \wedge h$ belongs to both y and Q, contradicting that y and Q are disjoint. This shows $y' \notin \bigcup_{j=1}^m U_{I_{\sigma j}}$.

We have produced two distinct points y', z' of the Stone space belonging to the open set $\phi(a)$ and not belonging to $\bigcup_{j=1}^m U_{I_{\sigma j}}$. This shows that x cannot be an isolated point of $U_{I_\sigma} - \bigcup_{j=1}^m U_{I_{\sigma j}}$. □

We are now able to prove our desired result.

Main Lemma *For each $m, n, k \geq 1$ there is an interior map from X onto $Q_{m,n,k}$.*

Proof Consider the map $f : X \to T_{m,n}$ given by Proposition 4.8. For σ of length at most $n-1$, by Lemma 4.9, the set $U_{I_\sigma} - \bigcup_{j=1}^m U_{I_{\sigma j}}$ has no isolated points in the subspace topology, and if σ has length n we have U_{I_σ} is open so trivially has no isolated points as X has none. So for each $\sigma \in T_{m,n}$ we have $f^{-1}(\sigma)$ has no isolated points, and as each $f^{-1}(\sigma)$ is locally compact and Hausdorff, it is k-resolvable (see, e.g., [9, p. 332]). This means we can split $f^{-1}(\sigma)$ into k disjoint pieces $C_1^\sigma, \ldots, C_k^\sigma$ so that every open subset of X that intersects $f^{-1}(\sigma)$ non-trivially intersects each of these sets non-trivially. Define $g : X \to Q_{m,n,k}$ by mapping all elements in C_i^σ to the i^{th} element q_i^σ of the cluster associated with σ. Clearly g is onto. For an open $U \subseteq X$, if U intersects $f^{-1}(\sigma)$ nontrivially, it intersects each C_i^σ nontrivially. It then follows by Proposition 4.8 that $g(U) = \{q_i^\sigma : \sigma \in f(U)\}$, so $g(U)$ is an upset, hence is open. Suppose U is an upset of $Q_{m,n,k}$. If U contains one element of a cluster, it contains all elements of the cluster. Then for $V = \{\sigma \in T_{m,n} : q_i^\sigma \in U \text{ for some } i \leq m\}$ we have $g^{-1}(U) = f^{-1}(V)$, so it is open in X. □

Corollary 4.10 *For each finite quasi-tree Q, there exists an interior map from X onto Q.*

Proof It follows from [4, Lem. 5] that for each finite quasi-tree Q there exist $m, n, k \geq 1$ such that Q is an interior image of $Q_{m,n,k}$. Then the composition $X \to Q_{m,n,k} \to Q$ is interior and onto. □

Now we are ready to establish our first main result.

Theorem 4.11 **S4** *is the modal logic of* $\mathbb{N}^*$.

Proof Since $\mathbb{N}^*$ is a topological space, every theorem of **S4** is satisfied in $\mathbb{N}^*$. If φ is not provable in **S4**, there exists a finite quasi-tree Q such that φ is refuted on Q. By [10, p. 95], $\mathbb{N}^*$ is homeomorphic to X. Thus, by Corollary 4.10, there exists an interior map from $\mathbb{N}^*$ onto Q. Finally, since validity of formulas is preserved by onto interior maps and φ is refuted on Q, it is also refuted on $\mathbb{N}^*$. Therefore, **S4** is complete with respect to $\mathbb{N}^*$. □

5 The modal logic of $\beta(\mathbb{N})$

Let **S4.1.2** denote the normal extension of **S4** by the axiom $\Box\Diamond p \leftrightarrow \Diamond\Box p$. In this section we show that **S4.1.2** is the modal logic of $\beta(\mathbb{N})$.

Let $\langle X, \leq\rangle$ be a quasi-ordered set. We call $x \in X$ a *maximal point* if $x \leq y$ implies $x = y$ for each $y \in X$. Let $\max X$ denote the set of maximal points of X. It is well-known (see, e.g., [6, pp. 80, 82]) that $\Box\Diamond p \to \Diamond\Box p$ is valid in $\langle X, \leq\rangle$ iff for each $x \in X$ there exists $y \in \max X$ with $x \leq y$, and that $\Diamond\Box p \to \Box\Diamond p$ is valid in $\langle X, \leq\rangle$ iff for each $x, y, z \in X$ with $x \leq y$ and $x \leq z$ there exists $w \in X$ such that $y \leq w$ and $z \leq w$. Therefore, if X is finite and rooted, then $\Box\Diamond p \leftrightarrow \Diamond\Box p$ is valid in X iff X has a top element. Moreover, it is well-known (see, e.g., [6, p. 144]) that **S4.1.2** is complete with respect to finite rooted quasi-ordered sets with a top element.

For a finite rooted quasi-ordered set $\langle X, \leq\rangle$ let $X^\top$ denote the quasi-ordered set obtained by adjoining $\top$ to X as the top element.

Lemma 5.1 *Let $\langle X, \leq\rangle$ be a finite rooted quasi-ordered set. If there is an interior map from $\mathbb{N}^*$ onto X, then there is an interior map from $\beta(\mathbb{N})$ onto $X^\top$.*

Proof Let f be an interior map from $\mathbb{N}^*$ onto X. Define $g : \beta(\mathbb{N}) \to X^\top$ by

$$g(x) = \begin{cases} \top & \text{if } x \in \mathbb{N}, \\ f(x) & \text{otherwise.} \end{cases}$$

Since f is onto, it is clear that g is a well-defined onto map. To see that g is continuous, let U be an upset of $X^\top$, and let $V = U - \{\top\}$. Clearly V is an upset of X. Moreover, $g^{-1}(U) = \mathbb{N} \cup f^{-1}(V)$, which is open in $\beta(\mathbb{N})$ since $f^{-1}(V)$ is open in the subspace topology on $\mathbb{N}^*$. Finally, to see that g is open, let U be a basic open in $\beta(\mathbb{N})$. Then $g(U) = f(U) \cup \{\top\}$, which is an upset in $X^\top$ because $f(U)$ is an upset in X. Therefore, g is interior and onto. □

Now we are ready to establish our second main result.

Theorem 5.2 **S4.1.2** *is the modal logic of* $\beta(\mathbb{N})$.

Proof It follows from [5, Prop. 2.1] that $\Box\Diamond p \to \Diamond\Box p$ is valid in a topological space X iff the set of dense subsets of X is a filter. In particular, if the set $\text{Iso}(X)$ of isolated points of X is dense in X, then $\Box\Diamond p \to \Diamond\Box p$ is valid in X. Also it follows from [8, Thm. 1.3.3] that $\Diamond\Box p \to \Box\Diamond p$ is valid in a topological space X iff X is extremally disconnected. Since $\text{Iso}(\beta(\mathbb{N})) = \mathbb{N}$ is dense in $\beta(\mathbb{N})$ and $\beta(\mathbb{N})$ is extremally disconnected, $\beta(\mathbb{N})$ validates every theorem of **S4.1.2**. Suppose φ is not provable in **S4.1.2**. Then there exists a finite rooted quasi-ordered set with a top element refuting φ. We can assume that it has the form $Q^\top$ for some finite quasi-tree Q. By Corollary 4.10, there exists an interior onto map $f : \mathbb{N}^* \to Q$. By Lemma 5.1, there exists an interior onto map $g : \beta(\mathbb{N}) \to Q^\top$. Therefore, φ is refuted on $\beta(\mathbb{N})$. Thus, **S4.1.2** is complete with respect to $\beta(\mathbb{N})$. □

6 Conclusions

In this paper we showed that under the assumption of $(\mathfrak{a} = 2^\omega)$, the modal logic of $\mathbb{N}^*$ is **S4**, and that of $\beta(\mathbb{N})$ is **S4.1.2**. It remains an open question whether the same is true in ZFC. We recently became aware of a paper by P. Simon [16] that may be of use in this matter.

In proving our main results, we constructed an interior map from $\mathbb{N}^*$ onto every finite quasi-tree, and then used completeness of **S4** with respect to finite quasi-trees and preservation of validity of modal formulas under interior images to obtain the desired completeness. It is well-known (see, e.g., [15, pp. 64–65]) that **S4** is complete with respect to the infinite binary tree T, and that **S4.1.2** is complete with respect to T adjoined with a top element. One might think that an alternative (even easier) way of proving completeness of **S4** with respect to $\mathbb{N}^*$, and that of **S4.1.2** with respect to $\beta(\mathbb{N})$ would be by constructing an interior map from $\mathbb{N}^*$ onto T. We show now that such a map does not exist. Let $\mathfrak{F}$ denote the relational frame $\langle \mathbb{N}, \leq \rangle$, where $\leq$ is the standard ordering of $\mathbb{N}$. By identifying the immediate successor nodes of each node of T, we obtain that $\mathfrak{F}$ is an interior image of T in the Alexandroff topologies associated with $\mathfrak{F}$ and T, respectively. We show that $\mathfrak{F}$ is not an interior image of $\mathbb{N}^*$, which, by the above, implies that T is not an interior image of $\mathbb{N}^*$. Suppose f is an interior map from $\mathbb{N}^*$ onto $\mathfrak{F}$. Since $\{\uparrow n : n \in \mathbb{N}\}$ is a strictly decreasing family of open subsets of the Alexandroff topology on $\mathfrak{F}$ with empty intersection, by continuity of f, $\{f^{-1}(\uparrow n) : n \in \mathbb{N}\}$ is a strictly decreasing family of open subsets of $\mathbb{N}^*$ with empty intersection. As clopens of $\mathbb{N}^*$ form a basis and f is open, we then can produce a strictly decreasing family $\{A_n : n \in \mathbb{N}\}$ of clopens of $\mathbb{N}^*$ with $f(A_n) = \uparrow n$. Therefore, $f(\bigcap A_n) \subseteq \bigcap f(A_n) = \bigcap \uparrow n = \emptyset$, which is a contradiction since $\bigcap A_n$ is nonempty by compactness of $\mathbb{N}^*$.

Since **S4** is a modal companion of the propositional intuitionistic logic **Int** and **S4.1.2** is a modal companion of the logic $\mathbf{KC} = \mathbf{Int} + (\neg p \vee \neg\neg p)$ of weak excluded middle (see, e.g., [6, p. 325]), our main results imply that **Int** is complete with respect to $\mathbb{N}^*$, and that **KC** is complete with respect to $\beta(\mathbb{N})$. Algebraically, this means that the variety of all Heyting algebras is generated by the Heyting algebra of open subsets of $\mathbb{N}^*$, and that the variety of Heyting algebras satisfying the Stone identity $\neg x \vee \neg\neg x = 1$ is generated by the Heyting algebra of open subsets of the Stone–Čech compactification of $\mathbb{N}$.

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