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REGULAR COMPLETIONS OF LATTICES

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Abstract. A variety of lattices admits a meet dense regular completion if every lattice in the variety can be embedded into a complete lattice in the variety by an embedding that is meet dense and regular (preserves existing joins and meets). We show that exactly two varieties of lattices admit a meet dense regular completion, the variety of one-element lattices and the variety of all lattices. This extends an earlier result of Harding [8] showing these are the only two varieties of lattices closed under MacNeille completions.

1. Introduction

A completion of a lattice L is a lattice embedding $f : L \to C$ where C is a complete lattice. A completion is regular if it preserves all existing joins and meets in L, meaning that for any $S \subseteq L$, if S has a join in L then $f(\bigvee S) = \bigvee f[S]$, and if S has a meet in L, then $f(\bigwedge S) = \bigwedge f[S]$. A completion is meet dense if each element of C is the meet of elements of the image of f. We say a variety V of lattices admits a meet dense regular completion if each $L \in V$ has meet dense regular completion $f : L \to C$ into a complete lattice $C \in V$. Our purpose is to prove the following.

Main Theorem. *The only varieties of lattices that admit a meet dense regular completion are the trivial variety of one-element lattices and the variety of all lattices.*

Our interest in this result is motivated by general considerations of completions of lattices with additional operations. There is a growing body of work describing

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when a variety of such algebras is closed under various methods of completion, such as the MacNeille completion [7, 9, 12, 13], or canonical completion [4, 6, 5, 11]. Generally, very few varieties are closed under MacNeille completions [2, 3, 8]. This is unfortunate as finding a regular completion is often of considerable interest, especially in matters related to algebraic logic [14], and the MacNeille completion is the primary technique to construct regular completions.

One would expect it to be possible for a variety to admit a regular completion, yet fail to be closed under MacNeille completions. Indeed this is the case, as was recently demonstrated for the variety generated by the three-element Heyting algebra [10]. Our intent is to explore this phenomenon, and as a first step, to see if there are any varieties of lattices that are not closed under MacNeille completions but do admit a regular completion. This remains an open problem, but we are able to establish our result under the additional assumption that the completions are also meet dense. We note that in the above example of the variety generated by the three element Heyting algebra, the regular completion obtained in [10] is also meet dense.

2. The Construction

Throughout, L is a lattice. For each $a \in L$ and each integer n we introduce formal symbols a_n and a'_n. We then set $P = \{a_n, a'_n \mid a \in L \text{ and } n \in \mathbb{Z}\}$ and let $F(P)$ be the free distributive lattice over the set P. We have frequent cause to use the following notation. Suppose S is a non-empty finite subset of L and for each $s \in S$ we have an integer m_s. Then $\bigwedge_S s_{m_s}$, $\bigvee_S s_{m_s}$, $\bigwedge_S s'_{m_s}$ and $\bigvee_S s'_{m_s}$ are elements of $F(P)$ representing certain meets and joins of generators.

Definition 1. Let θ be the smallest congruence on $F(P)$ so that for any integer n, any non-empty subsets S, T of L, and any families of integers m_s, n_t we have

(1) $a_n/\theta \leq a_{n+1}/\theta$,
(2) $a'_n/\theta \leq a'_{n+1}/\theta$,
(3) if $\bigwedge S \leq \bigvee T$, then $\bigwedge_S s_{m_s}/\theta \leq \bigvee_T t'_{n_t}/\theta$.

We then set $D(L) = F(P)/\theta$.

We call the elements $a_n/\theta, a'_n/\theta$ where $a \in L$ and $n \in \mathbb{Z}$ the generators of $D(L)$. Using distributivity, each element of $D(L)$ is a finite join of elements, each of which is a finite meet of generators; and each element of $D(L)$ is a finite meet of elements, each of which is a finite join of generators. Each element of $D(L)$ that is a finite meet of generators is of the following form:

$$(\bigwedge_{s\in S} s_{m_s} \wedge \bigwedge_{t\in T} t'_{n_t})/\theta$$

where S, T are finite subsets of L not both empty, and m_s, n_t are families of integers. Here the point is that one need not consider a meet with both a_m/θ and a_n/θ or with both a'_m/θ and a'_n/θ in view of the first two conditions of Definition 1. The dual statement holds for elements that are finite joins of generators.

The following is key to understanding the ordering of $D(L)$.

Lemma 2.1. *Let S, T, U, V be finite subsets of L with $S \cup T$ and $U \cup V$ non-empty, and m_s, n_t, p_u, q_v be families of integers. Then*

$$(\bigwedge_{s\in S} s_{m_s} \wedge \bigwedge_{t\in T} t'_{n_t})/\theta \leq (\bigvee_{u\in U} u_{p_u} \vee \bigvee_{v\in V} v'_{q_v})/\theta$$

if and only if at least one of the following conditions holds: (i) $s = u$ and $m_s \leq p_u$ for some $s \in S$ and $u \in U$, (ii) $t = v$ and $n_t \leq v_q$ for some $t \in T$ and $v \in V$, or (iii) S and V are non-empty and $\bigwedge S \leq \bigvee V$.

Proof. For convenience, set $x = \bigwedge_S s_{m_s} \wedge \bigwedge_T t'_{n_t}$ and $y = \bigvee_U u_{p_u} \vee \bigvee_V v'_{q_v}$. If any of these conditions holds, it follows from Definition 1 that $x/\theta \leq y/\theta$. Indeed, the first condition gives $s_{m_s}/\theta \leq u_{p_u}/\theta$, the second gives $t'_{n_t}/\theta \leq v'_{q_v}/\theta$, and the third gives $\bigwedge_S s_{m_s}/\theta \leq \bigvee_V v'_{q_v}/\theta$. For the converse assume that none of these conditions hold, and we show the inequality does not hold.

Define $\alpha : P \longrightarrow 2$ by setting

$$\alpha(z_n) = \begin{cases} 1 & \text{if } z \in S,\, n \geq m_z, \\ 0 & \text{else.} \end{cases} \qquad \alpha(z'_n) = \begin{cases} 0 & \text{if } z \in V,\, n \leq q_z, \\ 1 & \text{else.} \end{cases}$$

Then there is a homomorphism $\bar{\alpha} : F(P) \longrightarrow 2$ extending α. As the first condition doesn't hold, $\overline{\alpha}(y) = 0$, and as the second condition doesn't hold, $\overline{\alpha}(x) = 1$. Let $\phi = \ker \bar{\alpha}$. So $x/\phi \not\leq y/\phi$. We claim that ϕ is a congruence satisfying the conditions of Definition 1. From this it follows that $\theta \subseteq \phi$, hence $x/\theta \not\leq y/\theta$ as required.

For any $a \in L$ and any integer n we have $\overline{\alpha}(a_n) \leq \overline{\alpha}(a_{n+1})$ and $\overline{\alpha}(a'_n) \leq \overline{\alpha}(a'_{n+1})$. It follows that $a_n/\phi \leq a_{n+1}/\phi$ and $a'_n/\phi \leq a'_{n+1}/\phi$. This gives the first two conditions of Definition 1. For the third, let E, F be non-empty finite subsets of L with $\bigwedge E \leq \bigvee F$ and j_e, k_f be families of integers. We must show $\overline{\alpha}(\bigwedge_E e_{j_e}) \leq \overline{\alpha}(\bigvee_F f'_{k_f})$. If $E \not\subseteq S$, this follows as $\overline{\alpha}(\bigwedge_E e_{j_e}) = 0$. If $F \not\subseteq V$, this follows as $\overline{\alpha}(\bigvee_F f'_{k_f}) = 1$. The remaining possibility, that $E \subseteq S$ and $F \subseteq V$, cannot happen as this would imply S, V are non-empty and $\bigwedge S \leq \bigwedge E \leq \bigvee F \leq$

$\bigvee V$, contrary to our assumption that the above conditions do not hold. So ϕ satisfies the conditions of Definition 1. □

The following notational device is quite convenient. We use lower case letters such as a for elements of L, and the corresponding upper case letter for associated subsets of $D(L)$ as described below.

Definition 2. For any $a \in L$ set

(1) $A = \{a_n/\theta : n \in \mathbb{Z}\}$,
(2) $A' = \{a'_n/\theta : n \in \mathbb{Z}\}$.

Further, we use L and U for the set of lower bounds and the set of upper bounds of a set of elements. So UA is the set of upper bounds of A in $D(L)$, and so forth.

Note, for $a \in L$, each element of UA is a finite meet of elements, each of which is a finite join of generators and belongs to UA; and each element of LA' is a finite join of elements, each of which is a finite meet of generators and belongs to LA'.

Lemma 2.2. *Suppose $a, b \in L$.*

(1) *$A \subseteq LA'$,*
(2) *If $a \leq b$, then $UB \subseteq UA$.*

Proof. 1. From the third condition in Definition 1, we have $a_m/\theta \leq a'_n/\theta$ for each m, n as $\bigwedge\{a\} \leq \bigvee\{a\}$. 2. Suppose x is an upper bound of B that is a finite join of generators, say $x = (\bigvee_U u_{p_u} \vee \bigvee_V v'_{q_v})/\theta$. As $b_m/\theta \leq x$ for each integer m, by Lemma 2.1 we conclude V is non-empty and $b \leq \bigvee V$. From this it follows that $a_m/\theta \leq x$ for each m, hence x is an upper bound of A. It follows that every upper bound of B is an upper bound of A. □

Lemma 2.3. *Suppose $a, b \in L$.*

(1) *If $x \in UA$, then $x = \bigwedge\{x \vee y : y \in A'\}$,*
(2) *If $a \wedge b = c$ and $x \in C'$, then $x = \bigwedge\{x \vee (y \wedge z) : y \in A'$ and $z \in B'\}$.*

Proof. 1. It is enough to show that for $x \in UA$, if w is a lower bound of $\{x \vee y : y \in A'\}$, then $w \leq x$. We show this in the case that w is a finite meet of generators $(\bigwedge_S s_{m_s} \wedge \bigwedge_T t'_{n_t})/\theta$ and x is a finite join of generators $(\bigvee_U u_{p_u} \vee \bigvee_V v'_{q_v})/\theta$. As $w \leq x \vee a'_k/\theta$ for each integer k, applying Lemma 2.1, the only conditions that could hold for each k would imply directly that $w \leq x$, or give $\bigwedge S \leq \bigvee(V \cup \{a\})$. But x being an upper bound of A gives $a \leq \bigvee V$, so this condition too yields $w \leq x$. The case for general x, w is handled easily as x is a finite meet $x = x_1 \wedge \cdots \wedge x_m$ and w is a finite join $w = w_1 \vee \cdots \vee w_n$ where each x_i is a finite join of generators and each w_j is a finite meet of generators.

2. We must show that each lower bound w of $\{x \vee (y \wedge z) : y \in A', z \in B'\}$ satisfies $w \leq x$. We show this in the case that w is a finite meet of generators $(\bigwedge_S s_{m_s} \wedge \bigwedge_T t'_{n_t})/\theta$, the general case follows as an arbitrary lower bound is a finite join of lower bounds of this form. As w is a lower bound of this set, $w \leq x \vee (y \wedge z)$ for each $y \in A'$ and $z \in B'$. Using the distributivity of $D(L)$, we have $w \leq (x \vee y) \wedge (x \vee z)$, hence $w \leq x \vee y$ for each $y \in A'$ and $w \leq x \vee z$ for each $z \in B'$. As $x \in C'$ we have $x = c'_p/\theta$ for some integer p. Therefore $w \leq (c'_p \vee a'_q)/\theta$ for all integers q and $w \leq (c'_p \vee b'_q)/\theta$ for all integers q. Applying Lemma 2.1 to this first condition leads directly to the conclusion that $w \leq x$, or to the conclusion that $\bigwedge S \leq c \vee a = a$, and applying this lemma to the second condition we either obtain directly that $w \leq x$, or $\bigwedge S \leq c \vee b = b$. So either we obtain directly that $w \leq x$, or that $\bigwedge S \leq a \wedge b = c$, and this condition also implies $w \leq x$. □

We remark that the above lemmas depend only on the definition of θ, and this definition is symmetric with respect to meets and joins. So the dual versions of these results hold as well.

3. Main Theorem

We assume the lattice L and the distributive lattice $D(L)$ are as in the previous section. We assume also that $f : D(L) \longrightarrow Q$ is a meet dense and regular completion. Define $\varphi : L \longrightarrow Q$ by setting $\varphi(a) = \bigvee f[A]$.

Lemma 3.1. *If $a \in L$, then $\varphi(a) = \bigvee f[A] = \bigwedge f[UA] = \bigwedge f[A'] = \bigvee f[LA']$.*

Proof. By Lemma 2.2 we have $A' \subseteq UA$ so $\bigwedge f[UA] \leq \bigwedge f[A']$. Suppose $x \in UA$. By Lemma 2.3 $x = \bigwedge\{x \vee y : y \in A'\}$, hence by regularity $f(x) = \bigwedge\{f(x) \vee f(y) : y \in A'\}$. It follows that $\bigwedge f[A'] \leq f(x)$. As this holds for all $x \in UA$ we have $\bigwedge f[A'] \leq \bigwedge f[UA]$, hence equality. That $\bigvee f[A] = \bigvee f[LA']$ follows by duality. Finally, as f is meet dense, we have $\bigvee f[A] = \bigwedge f[UA]$. □

We remark that the final step in the above proof is the only place we use the meet density requirement, and that this could equally well be replaced join density, giving $\bigwedge f[A'] = \bigvee f[LA']$.

Theorem 3.2. *The map $\varphi : L \to Q$ is a lattice embedding.*

Proof. If $a \leq b$, by Lemma 2.2, $UB \subseteq UA$, so $\varphi(a) = \bigwedge f[UA] \leq \bigwedge f[UB] = \varphi(b)$. Thus φ is order preserving. Suppose $a \wedge b = c$. Then $\varphi(c) \leq \varphi(a) \wedge \varphi(b)$.

Suppose $x \in C'$. Then by Lemma 2.3, $x = \bigwedge\{x \vee (y \wedge z) : y \in A', z \in B'\}$, so by regularity $f(x)$ is equal to $\bigwedge\{f(x) \vee (f(y) \wedge f(z)) : y \in A', z \in B'\}$. By general principles, this meet is greater than or equal to $f(x) \vee \bigwedge\{f(y) \wedge f(z) : y \in A', z \in B'\}$, and it follows that $\bigwedge f[A'] \wedge \bigwedge f[B'] \leq f(x)$. As this holds for each $x \in C'$ we have $\bigwedge f[A'] \wedge \bigwedge f[B'] \leq \bigwedge f[C']$, so $\varphi(a) \wedge \varphi(b) \leq \varphi(c)$, hence equality. Showing that φ preserves finite joins follows by duality using the dual of Lemma 2.3. Finally, if $a \not\leq b$, then by Lemma 2.1 we have $a_0/\theta \not\leq b'_0/\theta$, and as f is an embedding this gives $f(a_0/\theta) \not\leq f(b'_0/\theta)$. This shows $\varphi(a) = \bigvee f[A] \not\leq \bigwedge f[B'] = \varphi(b)$. So φ is an embedding. $\square$

Main Theorem *The only varieties of lattices that admit a meet dense regular completion are the variety of one-element lattices and the variety of all lattices.*

Proof. Clearly each of these varieties admits such a completion, namely the MacNeille completion. To show that these are the only such varieties assume V is a variety containing a lattice with more than one element and that V admits a meet dense regular completion. Take any lattice L. As V contains the variety of distributive lattices, $D(L)$ belongs to V, therefore there is a meet dense regular completion $f : D(L) \to Q$ with $Q \in V$. By the previous theorem, the map $\varphi : L \to Q$ is a lattice embedding, so L is isomorphic to a sublattice of Q, hence L belongs to V. Thus V is the variety of all lattices. $\square$

4. Conclusion

It remains an open problem whether the only varieties of lattices admitting a regular completion are the variety of one-element lattices and the variety of all lattices. There are some things known about this. If V is a variety of lattices admitting a regular completion and containing a lattice with more than one element, then as V contains all distributive lattices, it follows from a result of Crawley [1] that V contains the five-element non-modular lattice. Using techniques similar to those in this paper, we are able to show a certain class of finite lattices must also belong to V. This class includes lattices such as M_3, the benzene ring, and so forth. However, these techniques seem quite far from establishing that V must be the variety of all lattices.

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