Type-2 Fuzzy Sets and Bichains

John Harding, Carol Walker and Elbert Walker

Abstract This paper is a continuation of the study of the variety generated by the truth value algebra of type-2 fuzzy sets. That variety and some of its reducts were shown to be generated by finite algebras, and in particular to be locally finite. A basic question remaining is whether or not these algebras have finite equational bases, and that is our principal concern in this paper. The variety generated by the truth value algebra of type-2 fuzzy sets with only its two semilattice operations in its type is generated by a four element algebra that is a bichain. Our initial goal is to understand the equational properties of this particular bichain, and in particular whether or not the variety generated by it has a finite equational basis.

1 Introduction

The underlying set of the algebra of truth values of type-2 fuzzy sets is the set $M = Map([0,1],[0,1])$ of all functions from the unit interval into itself. This set is equipped with the binary operations $+$ and $\cdot$, the unary operation *, and the nullary operations $\bar{1}$ and $\bar{0}$ as spelled out below, where $\vee$ and $\wedge$ denote maximum and minimum, respectively.

$$\begin{aligned}(f+g)(x) &= \sup\{f(y)\wedge g(z) : y\vee z = x\}\\ (f\cdot g)(x) &= \sup\{f(y)\wedge g(z) : y\wedge z = x\}\\ f^*(x) &= \sup\{f(y) : 1-y = x\} = f(1-x)\end{aligned}$$

John Harding
New Mexico State University, Las Cruces NM 88003, e-mail: jharding@nmsu.edu

Carol Walker
New Mexico State University, Las Cruces NM 88003, e-mail: hardy@nmsu.edu

Elbert Walker
New Mexico State University, Las Cruces NM 88003, e-mail: elbert@nmsu.edu

$$\bar{1}(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \end{cases} \quad \text{and} \quad \bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$$

The algebra of truth values of type-2 fuzzy sets was introduced by Zadeh in 1975, generalizing the truth value algebras of ordinary fuzzy sets, and of interval-valued fuzzy sets. (Sometimes in the fuzzy literature, the operations $+$ and $\cdot$ are denoted $\sqcup$ and $\sqcap$, respectively, but we choose to use the less cumbersome notations $+$ and $\cdot$. We also frequently write fg instead of $f \cdot g$.) The definitions of the convolutions $+$, $\cdot$, and $*$ are sometimes referred to as Zadeh's extension principle.

Definition 1. The algebra $\mathbb{M} = (M, +, \cdot, ^*, \bar{1}, \bar{0})$ is the **algebra of truth values** for fuzzy sets of type-2.

Type-2 fuzzy sets, that is, fuzzy sets with this algebra $\mathbb{M}$ of truth values, play an increasingly important role in applications, making $\mathbb{M}$ of some theoretical interest. See, for example, [3, 9, 13, 14, 15, 20].

We are concerned here with the equational properties of this algebra, much as one is concerned with the equational properties of the Boolean algebras used in classical logic. The main question we are interested in is whether there is a finite equational basis for the variety $\mathscr{V}(\mathbb{M})$ generated by $\mathbb{M}$. We have made some progress toward this, and other questions, but it remains open.

An important step in understanding the equational theory of $\mathbb{M}$ was taken in [4, 19] where the operations $+$ and $\cdot$ were written in a tractable way using the auxiliary operations L and R, where f^L and f^R are the least increasing and decreasing functions, respectively, above f. Using this, it was shown that $\mathbb{M}$ satisfies the following equations.

Proposition 1. *Let $f, g, h \in M$.*

1. $f + f = f$; $f \cdot f = f$
2. $f + g = g + f$; $f \cdot g = g \cdot f$
3. $f + (g + h) = (f + g) + h$; $f \cdot (g \cdot h) = (f \cdot g) \cdot h$
4. $f + (f \cdot g) = f \cdot (f + g)$
5. $\bar{1} \cdot f = f$; $\bar{0} + f = f$
6. $f^{**} = f$
7. $(f + g)^* = f^* \cdot g^*$; $(f \cdot g)^* = f^* + g^*$

Algebras, such as $\mathbb{M}$, that satisfy the above equations have been studied in the literature under the name *De Morgan bisemilattices* [1, 10, 11].

Definition 2. A **variety** of algebras is the class of all algebras of a given type satisfying a given set of identities (a **basis** for the variety). Equivalently (by a famous theorem of Birkhoff), a variety is a class of algebras of the same type which is closed under the taking of homomorphic images, subalgebras and (direct) products.

Definition 3. For an algebra $\mathbb{A}$, the **variety** $\mathscr{V}(\mathbb{A})$ **generated by** $\mathbb{A}$ is the class of all algebras with the same type as $\mathbb{A}$ that satisfy all the equations satisfied by $\mathbb{A}$. An algebra $\mathbb{A}$ is **locally finite** if each finite subset of $\mathbb{A}$ generates a finite subalgebra of $\mathbb{A}$, and a variety is **locally finite** if each algebra in the variety is locally finite.

An advance in understanding $\mathbb{M}$ and its equational properties came in [7], where it was shown that the variety $\mathscr{V}(\mathbb{M})$ is finitely generated, meaning it is generated by a single finite algebra. In fact, it is generated by the complex algebra (algebra of subsets) of a 5-element bounded chain with involution. In this same paper, it was shown $\mathscr{V}(\mathbb{M})$ is generated by a smaller 12-element De Morgan bisemilattice, but this algebra is not so easily described. An important consequence of this result is an algorithm to determine whether an equation holds in $\mathbb{M}$. One simply checks to see if the equation holds in the finite algebra generating $\mathscr{V}(\mathbb{M})$. In this same paper, a normal form for terms in $\mathscr{V}(\mathbb{M})$ was given, and used to develop a syntactic algorithm to determine when an equation holds in $\mathscr{V}(\mathbb{M})$.

It is natural to consider whether the equations in Proposition 1 could be a basis for the variety $\mathscr{V}(\mathbb{M})$; that is, whether or not every equation satisfied by the algebras in $\mathscr{V}(\mathbb{M})$ is a consequence of those equations in Proposition 1. This is not the case as $\mathscr{V}(\mathbb{M})$ is locally finite, and there are De Morgan bisemilattices that are not locally finite, such as certain ortholattices. So to find a basis for the variety $\mathscr{V}(\mathbb{M})$ one must add equations to this list. We will exhibit later some equations that hold in $\mathbb{M}$ that are not consequences of the equations above. Whether there is a finite basis for $\mathscr{V}(\mathbb{M})$ remains open.

The observant reader at this point will have considered Baker's Theorem [2] that says a finitely generated congruence distributive variety has a finite basis. Unfortunately we cannot apply this result as $\mathscr{V}(\mathbb{M})$ is not congruence distributive, as is noted in a later section.

We decided to simplify the problem, and restrict attention to equations involving only the operations $+$ and $\cdot$ and not using the negation $*$ or constants $\bar{1}$ and $\bar{0}$.

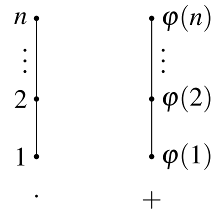
Definition 4. An algebra $(A,\cdot,+)$ with two binary operations is called a **bisemilattice** if it satisfies equations 1 – 3 of Proposition 1, and a **Birkhoff system** if it satisfies equations 1 – 4 of Proposition 1.

Of course the reduct $(M,+,\cdot)$ of $\mathbb{M}$ to this type satisfies equations 1 – 4 of Proposition 1, so is a Birkhoff system.

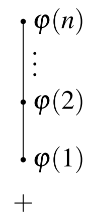
In any bisemilattice $(A,\cdot,+)$, the binary operations $\cdot$ and $+$ induce partial orders by $x \leq_{\cdot} y$ if $x = xy$ and $x \leq_{+} y$ if $x + y = y$. It is not difficult to show that these two partial orders are the same if and only if the bisemilattice is a lattice.

Definition 5. A bisemilattice $(A,\cdot,+)$ is a **bichain** if the two partial orders $\leq_{\cdot}$ and $\leq_{+}$ are chains.

A bichain is thus given by a set and two linear orderings on it. This is the same as giving an ordering on a set, and a permutation on that set. Of particular importance here will be finite bichains. Here we often assume the underlying set is $\{1,\ldots,n\}$, that the $\cdot$-ordering is $1 <_{\cdot} 2 <_{\cdot} \cdots <_{\cdot} n$, and that the $+$-ordering is given by some permutation φ of $\{1,2,\ldots,n\}$. The situation is shown below.



Any permutation φ gives an ordering of $1, 2, \ldots, n$ for the $+$-order, so up to isomorphism there are $n!$ n-element bichains. We assume the $\cdot$-order is $1 < 2 < \cdots < n$ and then just give the $+$-order. So we may depict bichains in the following manner:



Our reduct $(M, \cdot, +)$ is a Birkhoff system. Of course, the variety generated by this algebra is generated by the reduct of the 12-element De Morgan system that generates $\mathscr{V}(\mathbb{M})$, but one can do better. In [7] it was shown that the variety generated by $(M, +, \cdot)$ is generated by the 4-element bichain we call $\mathbb{B}$, shown in Figure 1.

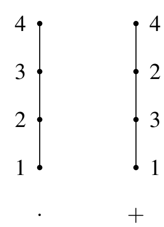
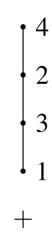


Fig. 1 The 4-element bichain $\mathbb{B}$

Of course, this bichain can be depicted simply by



While there is considerable literature on bisemilattices (see, for example, [11, 17, 18]), there seems to be relatively little known about the quite natural case of bichains. Our efforts here are largely devoted to studying bichains and the varieties they generate. We believe this is of interest for its own sake, as well as for its application to understanding equational properties of $\mathbb{M}$. One thing it enables us to do is to produce equations satisfied by $\mathbb{M}$ that are not a consequence of the equations 1 - 4 of Proposition 1. We list four such equations below. The names come from their donations by Fred (L)inton, Peter (J)ipsen, Keith (K)earnes, and a key equation (S) that is a splitting equation of a certain variety.

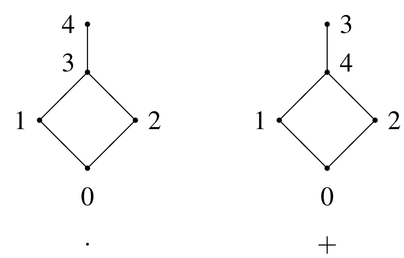
$$xz+y(x+z)=(x+z)(y+xz) \tag{L}$$

$$y(x+xz)=y(x+y)(x+z) \tag{J}$$

$$x(y+z)(xy+xz)=x(y+z)+(xy+xz) \tag{S}$$

$$x(xy+xz)=xy+xz \tag{K}$$

These equations hold in $\mathbb{B}$ as is easily checked. However, they do not hold in the variety of Birkhoff systems, so are not consequences of equations 1 - 4 of Proposition 1. The first three equations fail in the 3-element bichain denoted $\mathbb{A}_5$ in Figure 2 of the following section. The fourth is valid in all six 3-element bichains. Each subset of a bichain is a subalgebra, and it follows that this fourth equation (K) is valid in all bichains; however, it fails in the Birkhoff system depicted below.



We further remark that using the third equation (S) and several equations valid in all bichains, such as (K), we can prove the first two equations (L) and (J). Rather, a software package called Prover9 [12] can prove them. We conjecture that any equation valid in $\mathbb{B}$ can be proved from (S) and equations valid in *BiCh*, or equivalently, that $\mathscr{V}(\mathbb{B})$ is defined by the equations defining *BiCh* and the equation (S).

2 Subvarieties of $\mathscr{V}(\mathbb{B})$

Let *BiSemi* be the variety of all bisemilattices, *Birk* be the variety of all Birkhoff systems, *BiCh* be the variety generated by all bichains, *DL* be the variety of all distributive lattices, and *SL* be the variety of all bisemilattices satisfying $x \cdot y = x + y$, which is called the variety of semilattices. For any bisemilattice $\mathbb{S}$ we let $\mathscr{V}(\mathbb{S})$ be the variety generated by $\mathbb{S}$.

Proposition 2. *Every bichain is a Birkhoff system, so* $BiCh \subseteq Birk$.

Proof. Suppose x, y are elements of a bichain. Then each of xy and $x+y$ is either x or y, and we check that in the four possible cases $x(x+y) = x+xy$. □

The inclusion $BiCh \subseteq Birk$ is proper, since (K) is valid in all bichains, but not in all Birkhoff systems.

Below we describe and name all bichains with two or three elements.

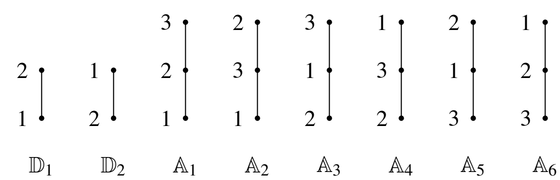


Fig. 2 The 2- and 3-element bichains

Note that $\mathbb{D}_1$ and $\mathbb{A}_1$ are distributive lattices so generate the variety *DL*, and $\mathbb{D}_2$ and $\mathbb{A}_6$ are semilattices so generate *SL* [2]. By [16] the join of *DL* and *SL* is the variety of distributive bisemilattices; that is, bisemilattices satisfying both distributive laws. As $\mathbb{D}_1$ and $\mathbb{D}_2$ are subalgebras of $\mathbb{A}_4$, and $\mathbb{A}_4$ is a quotient of their product, $\mathbb{A}_4$ generates $DL \vee SL$. By [11] the variety of bisemilattices satisfying the meet-distributive law $x(y+z) = xy+xz$ covers the distributive bisemilattices, as does the variety of bisemilattices satisfying the join-distributive law $x + yz = (x+y)(x+z)$. As $\mathbb{A}_2$ satisfies meet-distributivity but not join distributivity, and $\mathbb{A}_3$ satisfies join distributivity but not meet distributivity, $\mathscr{V}(\mathbb{A}_2)$ and $\mathscr{V}(\mathbb{A}_3)$ cover $\mathscr{V}(\mathbb{A}_4)$. As $\mathbb{A}_2$ and $\mathbb{A}_3$ are subalgebras of $\mathbb{B}$, we have $\mathscr{V}(\mathbb{A}_2) \vee \mathscr{V}(\mathbb{A}_3)$ is contained in $\mathscr{V}(\mathbb{B})$. Using the Universal Algebra calculator [5] we can find an equation to show this containment is strict. The algebras $\mathbb{A}_2$ and $\mathbb{A}_3$ satisfy

$$(x+z)(wx+w+y) = (x+z)(xy+w+y)$$

and this equation fails in $\mathbb{B}$. The program also provides equations to show neither $\mathscr{V}(\mathbb{A}_2)$ nor $\mathscr{V}(\mathbb{A}_3)$ are contained in $\mathscr{V}(\mathbb{A}_5)$, and $\mathscr{V}(\mathbb{A}_5)$ is not contained in $\mathscr{V}(\mathbb{B})$.

$$z(x+z)(y+z) = z(z+xy)$$
$$z + xz + yz = z + z(x+y)$$
$$x(y+z)(xy+xz) = x(y+z) + (xy+xz)$$

The first holds in $\mathbb{A}_5$ and fails in $\mathbb{A}_2$, the second holds in $\mathbb{A}_5$ and fails in $\mathbb{A}_3$, and the third holds in $\mathbb{B}$ and fails in $\mathbb{A}_5$. A diagram of the containments between these varieties follows.

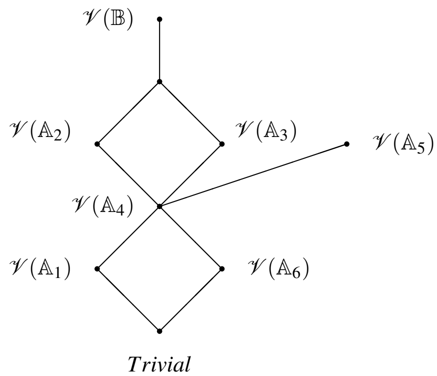


Fig. 3 Containments among varieties

Our conjecture is that $\mathscr{V}(\mathbb{B})$ is the largest subvariety of *BiCh* not containing $\mathbb{A}_5$, a situation known as a **splitting**. If this is indeed the case, $\mathscr{V}(\mathbb{B})$ is defined by a single equation called a **splitting equation**, together with equations defining *BiCh*. In this case, a splitting equation is

$$x(y+z)(xy+xz) = x(y+z)+(xy+xz) \tag{S}$$

That (S) is the splitting equation of $\mathbb{A}_5$ in *BiCh* comes through the fact that $\mathbb{A}_5$ is weakly projective in this variety, a topic we shall return to later. We remark that (S) is a type of generalized distributive law, with the left side of (S) being the meet of the two sides of the usual distributive law, and the right side of (S) being their join. We have not yet determined an equational basis of *BiCh*, and indeed do not even know if this variety is finitely based.

3 Bichains in the variety $\mathscr{V}(\mathbb{B})$

To lend some credence to our conjecture that $\mathscr{V}(\mathbb{B})$ is the largest subvariety of *BiCh* not containing $\mathbb{A}_5$, we use this section to show that a bichain belongs to $\mathscr{V}(\mathbb{B})$ if and only if it does not contain $\mathbb{A}_5$ as a subalgebra. We remark that if the variety *BiCh* were congruence distributive, our conjecture would follow from this using Jónsson's Lemma and Łoś's Theorem.

Theorem 1. *For a bichain $\mathbb{C}$, the following are equivalent.*

1. $\mathbb{C} \in \mathscr{V}(\mathbb{B})$.
2. $\mathbb{A}_5$ *is not a subalgebra of* $\mathbb{C}$.
3. $\mathbb{C}$ *satisfies (S).*

Proof. (1 $\Rightarrow$ 3) This is of course simply a matter of checking that the equation (S) holds in $\mathbb{B}$, but the situation is a bit more interesting than this. Note there is a congruence on $\mathbb{B}$ that collapses only the two middle elements $\{2,3\}$, and the resulting quotient is a distributive lattice. Take any equation $s=t$ that holds in all distributive lattices. If this equation is to fail in $\mathbb{B}$ for some choice of elements, it must be that s and t evaluate to 2 and 3. As $\{2,3\}$ is a subalgebra of $\mathbb{B}$ isomorphic to the 2-element semilattice, it then follows that $st=s+t$ holds in $\mathbb{B}$. The equation (S) is an instance of this, taking $s=t$ to be the meet distributive law.

(3 $\Rightarrow$ 2) Take $x=2$, $y=1$, and $z=3$ to see that $\mathbb{A}_5$ does not satisfy (S).

(2 $\Rightarrow$ 1) To show $\mathbb{C}\in\mathscr{V}(\mathbb{B})$, it is sufficient to show every finite sub-bichain of $\mathbb{C}$ belongs to $\mathscr{V}(\mathbb{B})$. Indeed, if $\mathbb{C}\notin\mathscr{V}(\mathbb{B})$, there is some equation valid in $\mathbb{B}$ that fails in $\mathbb{C}$. This equation involves only finitely many variables, so there is some finitely generated subalgebra of $\mathbb{C}$ that does not belong to $\mathscr{V}(\mathbb{B})$. But as $\mathbb{C}$ is a bichain, every subset of $\mathbb{C}$ is in fact a subalgebra of $\mathbb{C}$. So to show 2 $\Rightarrow$ 1, it is enough to show this for $\mathbb{C}$ a finite bichain.

We show by induction on $n=|\mathbb{C}|$ that if $\mathbb{A}_5$ is not isomorphic to a subalgebra of $\mathbb{C}$, then $\mathbb{C}\in\mathscr{V}(\mathbb{B})$. For $n\leq 3$ all n-element bichains are given in the figure in the previous section, and all but $\mathbb{A}_5$ are shown to belong to $\mathscr{V}(\mathbb{B})$. Suppose $\mathbb{C}$ has $n\geq 4$ elements. We first establish a lemma that handles several cases.

Lemma 1. *For a finite bichain* $\mathbb{C}$*, let* $\mathbb{C}\cup\{\infty\}$ *be the bichain formed from* $\mathbb{C}$ *by adding a new element to the bottom of the* $\cdot$*-order and the top of the* $+$*-order; let* $\mathbb{C}\cup\{b\}$ *be formed from* $\mathbb{C}$ *by adding a new element to the bottom of both orders; and let* $\mathbb{C}\cup\{t\}$ *be formed from* $\mathbb{C}$ *by adding a new element to the top of both orders. Then if* $\mathbb{C}\in\mathscr{V}(\mathbb{B})$*, so are* $\mathbb{C}\cup\{\infty\}$*,* $\mathbb{C}\cup\{b\}$*, and* $\mathbb{C}\cup\{t\}$*.*

Proof (Proof of Lemma). We first show $\mathbb{B}\cup\{\infty\}$, $\mathbb{B}\cup\{b\}$ and $\mathbb{B}\cup\{t\}$ belong to $\mathscr{V}(\mathbb{B})$. Note $\mathbb{B}\cup\{\infty\}$ is the quotient of $\mathbb{B}\times\mathbb{D}_2$ by the congruence θ that has one nontrivial block consisting of $\mathbb{B}\times\{1\}$; $\mathbb{B}\cup\{b\}$ is the subalgebra of $\mathbb{B}\times\mathbb{D}_1$ consisting of $\mathbb{B}\times\{2\}$ and $(1,1)$; and $\mathbb{B}\cup\{t\}$ is the subalgebra of $\mathbb{B}\times\mathbb{D}_2$ consisting of $\mathbb{B}\times\{1\}$ and $(4,2)$. As $\mathbb{D}_1$ and $\mathbb{D}_2$ belong to $\mathscr{V}(\mathbb{B})$, so do these algebras.

Assume $\mathbb{C}$ belongs to $\mathscr{V}(\mathbb{B})$. Then there is a set I, a subalgebra $\mathbb{S}\leq\mathbb{B}^I$, and an onto homomorphism $\varphi:\mathbb{S}\to\mathbb{C}$. Consider the constant function $\overline{\infty}$ in $(\mathbb{B}\cup\{\infty\})^I$ whose constant value is the new element ∞ added to $\mathbb{B}$. In $\mathbb{B}$, $x\cdot\infty=\infty$ and $x+\infty=\infty$. It follows that $\mathbb{S}\cup\{\infty\}$is a subalgebra of this power, and φ extends to a homomorphism from $\mathbb{S}\cup\{\infty\}$ onto $\mathbb{C}\cup\{\infty\}$. The arguments for $\mathbb{C}\cup\{b\}$ and $\mathbb{C}\cup\{t\}$ are similar, using powers of $\mathbb{B}\cup\{b\}$ and $\mathbb{B}\cup\{t\}$. □

(Proof of Theorem continued) Assume the $\cdot$-order of $\mathbb{C}$ is $1<2<\cdots<n$. If the bottom element of the $+$-order of $\mathbb{C}$ is 1, then $\mathbb{C}$ is isomorphic to $\mathbb{C}'\cup\{b\}$ where $\mathbb{C}'$ is the sub-bichain $\{2,\ldots,n\}$ of $\mathbb{C}$. Then by the inductive hypothesis and the above lemma, $\mathbb{C}\in\mathscr{V}(\mathbb{B})$. A similar argument handles the cases where either 1 or n is the top element of the $+$-order of $\mathbb{C}$. Set

$$U=\{k:2\leq k\leq n \text{ and } k \text{ precedes 1 in the } +\text{-order}\}$$

$$V = \{k : 2 \leq k \leq n \text{ and } 1 \text{ precedes } k \text{ in the } + \text{-order}\}$$

As 1 is not the bottom or top of the $+$-order, U and V are non-empty. Also, as $\mathbb{A}_5$ is not a subalgebra of $\mathbb{C}$, if $u \in U$ and $v \in V$, then $u < v$. Also, as n is not the top element of the $+$-order, V must have at least two elements. So there is some $2 \leq k \leq n-2$ with $U = \{2, \ldots, k\}$ and $V = \{k+1, \ldots, n\}$.

There are congruences θ and ϕ on $\mathbb{C}$ with θ collapsing $\{1, \ldots, k\}$ and nothing else, and ϕ collapsing V and nothing else. Note $\mathbb{C}/\theta$ is isomorphic to the sub-bichain $\{1, k+1, \ldots, n\}$ of $\mathbb{C}$, and $\mathbb{C}/\phi$ is isomorphic to the sub-bichain $\{1, \ldots, k, k+1\}$ of $\mathbb{C}$. It follows from the inductive hypothesis that $\mathbb{C}/\theta$ and $\mathbb{C}/\phi$ belong to $\mathscr{V}(\mathbb{B})$. As θ and ϕ intersect to the diagonal, $\mathbb{C}$ is a subalgebra of their product, so belongs to $\mathscr{V}(\mathbb{B})$. $\square$

At this point, if we had congruence distributivity, it would follow that every subdirectly irreducible in the variety *BiCh* is a bichain, and then the above theorem would imply $\mathscr{V}(\mathbb{B})$ is defined, relative to the equations defining *BiCh*, by the single equation (S). However we do not have congruence distributivity [17].

4 Splitting

In this section we investigate projectivity and splitting for various bichains, and in particular for $\mathbb{A}_5$. Our main result here shows there is a largest subvariety of *BiCh* not containing $\mathbb{A}_5$, and the theorem of the previous section leads us to believe this may be the variety $\mathscr{V}(\mathbb{B})$.

Definition 6. An algebra $\mathbb{P}$ is **weakly projective** in a variety $\mathscr{V}$ if for any two algebras $\mathbb{E}$ and $\mathbb{A}$ in $\mathscr{V}$, for every homomorphism $f : \mathbb{P} \to \mathbb{E}$, and for every onto homomorphism $g : \mathbb{A} \twoheadrightarrow \mathbb{E}$, there is a homomorphism $h : \mathbb{P} \to \mathbb{A}$ with $gh = f$.

The usual definition of projective uses the categorical notion of an epimorphism in place of the onto homomorphism g. In a variety $\mathscr{V}$, there may be more epimorphisms than onto homomorphisms, so an algebra that is weakly projective may not be projective. However, we do not know whether epimorphisms must be onto in either of the varieties *Birk* or *BiCh*.

The following well-known result [6] is a convenient reformulation.

Proposition 3. *An algebra $\mathbb{P}$ is weakly projective in $\mathscr{V}$ if and only if for every onto homomorphism $u : \mathbb{A} \twoheadrightarrow \mathbb{P}$, there is an embedding $r : \mathbb{P} \to \mathbb{A}$ with $u \circ r = \mathrm{id}_{\mathbb{P}}$.*

Weak projectives are of interest for several reasons, but our primary one lies in Proposition 4 below. Before stating this, we define for an algebra $\mathbb{P}$ in a variety $\mathscr{V}$,

$$\mathscr{W}(\mathbb{P}) = \{\mathbb{A} \in \mathscr{V} : \mathbb{P} \not\rightarrowtail \mathbb{A}\}$$

Here $\mathbb{P} \not\rightarrowtail \mathbb{A}$ means $\mathbb{P}$ is not isomorphic to a subalgebra of $\mathbb{A}$.

Proposition 4. *If $\mathbb{P}$ is weakly projective in $\mathscr{V}$ and subdirectly irreducible, then $\mathscr{W}(\mathbb{P})$ is a variety, and is the largest subvariety of $\mathscr{V}$ that does not contain $\mathbb{P}$.*

This is a well-known result [6] and not difficult to prove. The situation is sometimes referred to as a splitting, as it splits the lattice of subvarieties of $\mathscr{V}$ into two parts, those that contain the variety $\mathscr{V}(\mathbb{P})$, and those that are contained in $\mathscr{W}(\mathbb{P})$. Further, such a splitting yields an equation, called the splitting equation, defining the variety $\mathscr{W}(\mathbb{P})$ relative to the equations defining $\mathscr{V}$. We now apply these results in our setting.

Proposition 5. *The 2-element distributive lattice $\mathbb{D}_1$ is subdirectly irreducible and weakly projective in BiCh. Its splitting variety $\mathscr{W}(\mathbb{D}_1)$ is the variety SL of semilattices.*

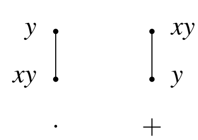
Proof. Clearly $\mathbb{D}_1$ is subdirectly irreducible (see Figure 2). Let $\mathbb{A}$ be a bichain and $f : \mathbb{A} \twoheadrightarrow \mathbb{D}_1$ be an onto homomorphism. Then there are x and y in $\mathbb{A}$ with $f(x) = 1$ and $f(y) = 2$. Then $f(xy) = 1$ and $f(x+y) = 2$, so xy is different from $x+y$. In any Birkhoff system we have $xy(x+y) = xy$ and $x+y+xy = x+y$. So there is a homomorphism $r : \mathbb{D}_1 \to \mathbb{A}$ defined by $r(1) = xy$ and $r(2) = x+y$, and this homomorphism satisfies $f \circ r = \mathrm{id}_{\mathbb{D}_1}$. So $\mathbb{D}_1$ is weakly projective.

To see that $\mathscr{W}(\mathbb{D}_1) = SL$, note that the two-element semilattice $\mathbb{D}_2$ belongs to $\mathscr{W}(\mathbb{D}_1)$, so one containment is trivial. For the other, suppose $\mathbb{A}$ does not belong to *SL*. Then there are $x, y \in \mathbb{A}$ with xy not equal to $x+y$, giving $\{xy, x+y\}$ is a subalgebra of $\mathbb{A}$ isomorphic to $\mathbb{D}_1$, so $\mathbb{A} \notin \mathscr{W}(\mathbb{D}_1)$. □

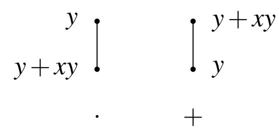
Note that for $\mathbb{D}_1$, these results hold also in the larger variety *Birk*.

Proposition 6. *The 2-element semilattice $\mathbb{D}_2$ is subdirectly irreducible and weakly projective in BiCh. Its splitting variety $\mathscr{W}(\mathbb{D}_2)$ in BiCh is the variety DL of distributive lattices.*

Proof. Clearly $\mathbb{D}_2$ is subdirectly irreducible (see Figure 2). Let $\mathbb{A}$ be a bichain and $f : \mathbb{A} \twoheadrightarrow \mathbb{D}_2$ be an onto homomorphism. Then there are x and y in $\mathbb{A}$ with $f(x) = 1$ and $f(y) = 2$. While we could now just jump to the answer, we build it step at a time to demonstrate an idea that will be used in a later proof. This same idea would have worked above. We first patch up the meet operation and consider the following:



We have $f(xy) = f(x)f(y) = (1)(2) = 1$ and $f(y) = 2$. Also $(xy)y = xy$. So $\{x, xy\}$ is a 2-element subset of $\mathbb{A}$ that works well with respect to meet. But it doesn't work well with respect to join since we would like that $y + xy = xy$ and there is no reason for this to be true. We work with what we have now and get it to work with respect to join.



Now by construction, this works well with respect to join, as $y+y+xy = y+xy$. It also works with respect to meet, as $y+xy = y(x+y)$, so $y(y+xy) = y+xy$. So $\{y+xy, y\}$ is a subalgebra of $\mathbb{A}$, $f(y+xy) = 2+1 = 1$ and $f(y) = 2$. So there is $r : \mathbb{D}_2 \to A$ with $r(1) = y+xy$ and $r(2) = y$, so $\mathbb{D}_2$ is weakly projective.

We next show that $\mathscr{W}(\mathbb{D}_2) = DL$. Surely $\mathscr{W}(\mathbb{D}_2) \supseteq DL$. To show $\mathscr{W}(\mathbb{D}_2) \subseteq DL$, suppose $\mathbb{A} \in BiCh$ and $\mathbb{A}$ has no subalgebra isomorphic to $\mathbb{D}_2$. Note that for any $x, y \in \mathbb{A}$ we have $x[x(x+y)] = x(x+y)$, and Birkhoff's equation $a(a+b) = a+ab$ gives $x+x(x+y) = x(x+x+y) = x(x+y)$. As $\mathbb{A}$ has no subalgebra isomorphic to $\mathbb{D}_2$, it follows that $x(x+y) = x$ for each $x, y \in \mathbb{A}$, and then by Birkhoff's equation that $x+xy = x$ for each $x, y \in \mathbb{A}$. So $\mathbb{A}$ is a lattice.

Consider the equations

$$x(x+y)(xz+y) = x(x+y)(xz+y+z)$$
$$z(x+y)(y+xz) = z(x+y)(y+z+xz)$$

Both hold in every bichain. To see this, as these equations involve three variables it is enough to check them in each 3-element bichain, and this is not difficult. So these equations hold in the variety *BiCh*, hence also in $\mathbb{A}$. The first does not hold in the 5-element modular, non-distributive lattice $\mathbb{M}_3$, and the second does not hold in the 5-element non-modular lattice $\mathbb{N}_5$. So $\mathbb{A}$ is a lattice containing neither $\mathbb{M}_3$ nor $\mathbb{N}_5$ as a subalgebra, showing $\mathbb{A}$ is a distributive lattice [2]. □

Note that our proof shows more. The algebra $\mathbb{D}_2$ is weakly projective in the larger variety *Birk*. It therefore has a splitting variety in *Birk*, but this is not *DL*, but the variety *Lat* of all lattices. This proof also shows $Lat \cap BiCh = DL$. In particular,

Corollary 1. *Any lattice in* $\mathscr{V}(\mathbb{B}) = \mathscr{V}(\mathbb{M})$ *is distributive.*

Now to the result most pertinent to our variety $\mathscr{V}(\mathbb{B})$. For convenience, we recall what $\mathbb{A}_5$ looks like.

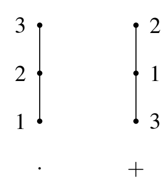
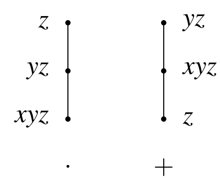


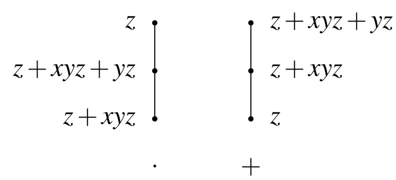
Fig. 4 The bichain $\mathbb{A}_5$

Proposition 7. $\mathbb{A}_5$ *is subdirectly irreducible and weakly projective in BiCh.*

Proof. The bichain $\mathbb{A}_5$ is subdirectly irreducible with its minimal congruence being the one collapsing 1 and 2. To see that it is weakly projective, assume $\mathbb{A} \in BiCh$ and $f : \mathbb{A} \twoheadrightarrow \mathbb{A}_5$. Then there are x, y, and z in $\mathbb{A}$ with $f(x) = 1$, $f(y) = 2$, and $f(z) = 3$. We follow the process in the previous proof to try to build a subalgebra of $\mathbb{A}$ that is isomorphic to $\mathbb{A}_5$. As our first step, we fix meets.



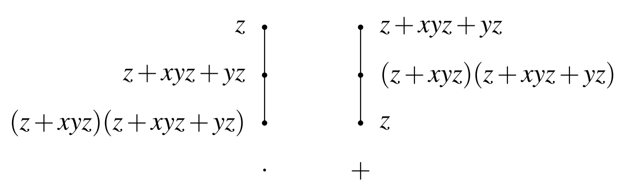
So now meets are okay, but joins are a problem. We fix them, bearing in mind we may wreck our meets when we do so.



So now we have fixed joins, but have troubles with the meets again. Before we continue further, note that Birkhoff's identity $a(a+b) = a+ab$ gives the following

$$\begin{aligned} z(z+xyz+yz) &= z+z(yz+xyz) \\ &= z+zyz(yz+x) \\ &= z+yz(yz+x) \\ &= z+yz+xyz \end{aligned}$$

So in fixing meets again, we may leave intact the top two elements of the $\cdot$-order to obtain the following.



Birkhoff's identity gives $(z+xyz)(z+xyz+yz) = z+xyz+yz(z+xyz)$. So the join of the bottom two elements of the $+$-order are correct since z will be absorbed when added to this element. To see the join of the top two elements of the $+$-order are correct, we again use Birkhoff's identity.

$$\begin{aligned} z+xyz+yz+(z+xyz+yz)*z+xyz) &= (z+xyz+yz)(z+xyz+yz+z+xyz) \\ &= (z+xyz+yz)(z+xyz+yz) \\ &= z+xyz+yz \end{aligned}$$

So after the last round of fixing meets, joins also are fixed.

We then get that $\{z, z+xyz+yz, (z+xyz)(z+xyz+yz)\}$ is a subalgebra of $\mathbb{A}$. One easily sees that $f((z+xyz)(z+xyz+yz)) = 1, f(z+xyz+yz) = 2$, and $f(z) = 3$. So $\mathbb{A}_5$ is weakly projective. □

We have shown somewhat more, that $\mathbb{A}_5$ is weakly projective in the larger variety *Birk*. In [8] we are able to extend this result significantly and show any finite bichain not containing the algebra $\mathbb{A}_4$ of Figure 2 is weakly projective in the variety *Birk*. However, it is the specific instance given above that is applicable to our study of Type-2 fuzzy sets. The main points are summarized below.

Theorem 2. *The algebra $\mathbb{A}_5$ is subdirectly irreducible and weakly projective in the variety BiCh. Its splitting variety $\mathscr{W}(\mathbb{A}_5)$ in BiCh contains $\mathscr{V}(\mathbb{B})$ and these two varieties contain exactly the same bichains. Equations defining $\mathscr{W}(\mathbb{A}_5)$ are given by the equations defining the variety BiCh and the splitting equation below, which is a generalized form of the distributive law.*

$$x(y+z)(xy+xz) = x(y+z)+(xy+xz). \tag{S}$$

Proof. That $\mathbb{A}_5$ is subdirectly irreducible and weakly projective in the variety *BiCh* is the content of Proposition 7. By Proposition 4, $\mathscr{W}(\mathbb{A}_5)$ is a variety and is the largest subvariety of *BiCh* not containing $\mathbb{A}_5$. From its definition, $\mathbb{B}$ belongs to $\mathscr{W}(\mathbb{A}_5)$, so $\mathscr{W}(\mathbb{A}_5)$ contains $\mathscr{V}(\mathbb{B})$. That $\mathscr{W}(\mathbb{A}_5)$ and $\mathscr{V}(\mathbb{B})$ contain the same bichains is provided by Theorem 1.

It remains to find the splitting equation for $\mathbb{A}_5$ in *BiCh*. Let F be the free Birkhoff system on the generators x, y, and z and let $\varphi : F \to \mathbb{A}_5$ be the homomorphism mapping x, y and z to $1, 2$ and 3, respectively. In the proof of the previous result, we found that $\{(z+yz+xyz)(z+xyz), z+yz+xyz, z\}$ is a subalgebra of F mapping isomorphically onto $\mathbb{A}_5$. Since $\{1,2\}$ generates the smallest non-trivial congruence on the subdirectly irreducible algebra $\mathbb{A}_5$, it follows from general considerations that the elements of this subalgebra mapped to 1 and 2 give the splitting equation for $\mathbb{A}_5$ in the variety of Birkhoff systems:

$$(z+yz+xyz)(z+xyz) = z+yz+xyz \tag{T}$$

Using the software packages Prover9 and Mace4 [12], we can find an example to show equation (T) is not equivalent to (S) in the variety of Birkhoff systems. However, consider the equations

$$x(x+y)(xz+y) = x(x+y)(xz+y+z) \tag{1}$$
$$x(xy+xz) = xy+xz \tag{2}$$

Considering cases, one checks that these equations are valid in every bichain, so are valid in the variety *BiCh*. Prover9 shows that in the presence of the identities for Birkhoff systems, equations (T), (1), and (2) together imply (S), and (S), (1), and (2) together imply (T). So in the variety *BiCh* we have (T) and (S) are equivalent, showing (S) is the splitting equation for S in the variety *BiCh*. □

We remark that as (S) is not equivalent to the splitting equation for $\mathbb{A}_5$ in the variety *Birk*, the splitting variety for $\mathbb{A}_5$ in *Birk* is strictly larger than the splitting variety for $\mathbb{A}_5$ in *BiCh*, and therefore strictly larger than $\mathscr{V}(\mathbb{B})$. So $\mathscr{V}(\mathbb{B})$ is not simply defined by the equations for Birkhoff systems plus (S), thus we do need some additional equations.

5 Conclusions and remarks

From a previous paper [7], we know that the variety generated by the truth value algebra of type-2 fuzzy sets with only its two semilattice operations in its type is generated by a 4-element algebra $\mathbb{B}$ that is a bichain and, in particular, a Birkhoff system.

Our aim is to find an equational basis for the variety generated by $\mathbb{B}$. This problem is difficult, but we have some progress. Our technique is to consider a particular 3-element bichain $\mathbb{A}_5$, show it is subdirectly irreducible and weakly projective, hence splitting, and that its splitting variety $\mathscr{W}(\mathbb{A}_5)$ in *BiCh* contains $\mathscr{V}(\mathbb{B})$.

Birk
|
BiCh
|
$\mathscr{W}(\mathbb{A}_5)$
|
$\mathscr{V}(\mathbb{B})$

We conjecture that $\mathscr{W}(\mathbb{A}_5) = \mathscr{V}(\mathbb{B})$. If so, this will show the splitting equation (S) for $\mathbb{A}_5$ then defines $\mathscr{V}(\mathbb{B})$ within *BiCh*. The results of Section 3 lend credence to this as we have shown a bichain belongs to $\mathscr{W}(\mathbb{A}_5)$ if and only if it belongs to $\mathscr{V}(\mathbb{B})$.

There remain a number of open problems in connection with this work. These include determining whether or not $\mathscr{W}(\mathbb{A}_5) = \mathscr{V}(\mathbb{B})$, and finding an equational basis for *BiCh*. Together, these will provide an equational basis for $\mathscr{V}(\mathbb{B})$, and hence for the $\cdot, +$ fragment of the truth value algebra $\mathbb{M}$ of type-2 fuzzy sets. One could conjecture that an equational basis for M with all its operations is one for $\mathbb{B}$ plus the equations for negation and the constants.

We have determined [8] that a bichain is weakly projective in the variety *Birk* if and only if it does not contain a copy of the bichain $\mathbb{A}_4$. As each weakly projective subdirectly irreducible algebra gives a splitting of the lattice of subvarieties, this adds to our knowledge of the lattice of subvarieties of Birkhoff systems, and in particular, of subvarieties of *BiCh*. We believe this variety *BiCh* is natural and of interest independent of its connection to fuzzy logic.

Finally, we remark that in preparing this still incomplete work, we made use of Universal Algebra Calculator [5], as well as the programs Prover9 and Mace4

[12] to find and work with equations. After finding equations with these programs we further verified all properties by hand. We are grateful to several people for providing equations of help to us, including Peter Jipsen, Keith Kearnes, and Fred Linton, and also to Anna Romanowska for several communications.

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