DYNAMICS IN THE DECOMPOSITIONS APPROACH TO QUANTUM MECHANICS

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ABSTRACT. In [10] it was shown that the direct product decompositions of any non-empty set, group, vector space, and topological space X form an orthomodular poset $\operatorname{Fact} X$. This is the basis for a line of study in foundational quantum mechanics replacing Hilbert spaces with other types of structures. The theory of states, observables, and probabilities in the context of decompositions was developed in [11, 12], and [13, 14] made ties to the categorical approach to quantum mechanics of Abramsky and Coecke [1] that focuses on compound systems. Here we develop dynamics and an abstract version of a time independent Schrödinger's equation in the setting of decompositions by considering representations of the group of real numbers in the automorphism group of the orthomodular poset $\operatorname{Fact} X$ of decompositions.

1. Introduction

In [10] it was shown that for many types of structures X, such as non-empty sets, groups, vector spaces, and topological spaces, the collection of direct product decompositions of X form an orthomodular poset $\operatorname{Fact} X$. When applied to a Hilbert space $\mathcal{H}$, this construction $\operatorname{Fact} \mathcal{H}$ yields the orthomodular lattice of closed subspaces of $\mathcal{H}$. Gleason's theorem and the spectral theorem describe states and observables in terms of $\operatorname{Fact} \mathcal{H}$, and this is generalized in [13, 14] to provide a treatment of states, observables, and probabilities in the setting of decompositions $\operatorname{Fact} X$ of more general types of structures.

To continue the development of a version of quantum mechanics based on decompositions, we describe here an approach to dynamics and an abstract version of a time independent Schrödinger's equation in the setting of $\operatorname{Fact} X$. This is based on representations of the group of real numbers in the automorphism group of $\operatorname{Fact} X$. In doing so, we develop much of the basics of general group representations in the setting of $\operatorname{Fact} X$.

This paper is arranged in the following manner. The second section provides preliminaries on $\operatorname{Fact} X$ in the setting of sets. The third section generalizes the construction of $\operatorname{Fact} X$ for sets to apply to a number of other settings to produce an orthoalgebra $\operatorname{Fact} X$ from the decompositions of more general types of structures X. While our aim is not to provide a version of categorical quantum mechanics, category theory is used as a tool here to conveniently present a wide array of instances of this construction. Group representations in sets and other structures are discussed in Section 4, and it is shown that the decompositions of a representation X again provide an orthoalgebra $\operatorname{Fact} X$. In the fifth section we review the treatment in [12] of observables, states, and probabilities in the setting of $\operatorname{Fact} X$. This is then used in the sixth section to give dynamics and a generalized time independent Schrödinger equation using representations of the reals and an energy observable. Here it is shown that the standard approach to dynamics in

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the Hilbert space setting is an instance of the general one. The seventh section makes steps towards classifying representations of the reals in sets and in Hausdorff spaces, and the final section makes some concluding remarks.

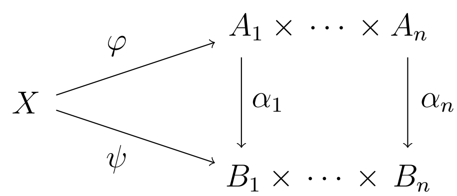
2. The decompositions Fact X of a set

In this section we provide the basics of a construction Fact X introduced in [10] to produce an orthomodular poset from the direct product decompositions of a set X. In the following section, extensions of this construction [10, 13, 14] are described that apply to a wide range of settings. These include universal algebras in the sense of [5], topological and uniform spaces, and objects of suitable types of categories.

Definition 2.1. *For a natural number n, an n-ary direct product decomposition of a set X consists of sets $A_1, \ldots, A_n$ and a bijection $\varphi : X \to A_1 \times \cdots \times A_n$.*

When $n = 2$ a direct product decomposition is called a binary decomposition, and when $n = 3$ a ternary decomposition. We frequently refer to a direct product decomposition simply as a decomposition. For any given $n \geq 1$ there is a proper class of n-ary decompositions of X. To define an equivalence relation on these decompositions, we note that for maps $\alpha_i : A_i \to B_i$ for $i = 1, \ldots, n$ there is an obvious map $\alpha_1 \times \cdots \times \alpha_n$ from $A_1 \times \cdots \times A_n$ to $B_1 \times \cdots \times B_n$.

Definition 2.2. *For a given n, two n-ary direct product decompositions $\varphi : X \to A_1 \times \cdots \times A_n$ and $\psi : X \to B_1 \times \cdots \times B_n$ are equivalent if there are bijections $\alpha_i : A_i \to B_i$ for each $i \leq n$ making the following diagram commute.*



The equivalence class of a decomposition φ is written $[\varphi]$.

The key to putting structure on the collection of equivalence classes of binary decompositions of a set X is to form new decompositions from old. A binary decomposition $\varphi : X \to A_1 \times A_2$ produces in an obvious way a new decomposition $\varphi' : X \to A_2 \times A_1$, and from a ternary decomposition $\gamma : X \to A_1 \times A_2 \times A_3$ we can build several binary decompositions, including

$$\gamma_{\{1\}\{23\}} : X \to A_1 \times (A_2 \times A_3) \quad \text{and} \quad \gamma_{\{12\}\{3\}} : X \to (A_1 \times A_2) \times A_3$$

Finally, we let $\{*\}$ denote some arbitrarily chosen 1-element set, and note that for any set X there are unique binary decompositions $\iota : X \to X \times \{*\}$ and $\iota' : X \to \{*\} \times X$.

Definition 2.3. *For a non-empty set X let* Fact X *be the set of all equivalence classes of binary decompositions of X. Define on* Fact X *two constants 1 and 0, a unary operation $\perp$ and a binary relation $\leq$ as follows.*

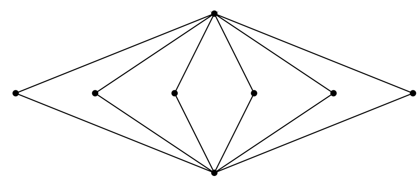
(1) 1 *is* $[\iota]$ *and* 0 *is* $[\iota']$.
(2) $[\varphi]^\perp = [\varphi']$
(3) $[\varphi] \leq [\psi]$ *iff there is a ternary decomposition γ with* $[\varphi] = [\gamma_{\{1\}\{23\}}]$ *and* $[\psi] = [\gamma_{\{12\}\{3\}}]$

Remark 2.4. We restrict attention to non-empty sets since for any set Y there is a unique bijection $\gamma : \emptyset \to \emptyset \times Y$. So there is a proper class of equivalence class of decompositions of $\emptyset$. More problematic, while the above structure could be defined on $\mathrm{Fact}\,\emptyset$, it is not well behaved.

We assume the reader is familiar with the notions of an orthomodular lattice (abbrev. OML), orthomodular poset (abbrev. OMP), and an orthoalgebra (abbrev. OA). See for instance [6, 28]. The following was established in [10].

Theorem 2.5. *For X a non-empty set,* $\mathrm{Fact}\,X$ *is an* OMP.

It is useful to give a small example. Consider a 4-element set $X = \{a, b, c, d\}$. This can be written as the product of a 1-element set and a 4-element set, as the product of a 4-element set and a 1-element set, and as the product of two 2-element sets. There are however six non-equivalent decompositions φ of X as the product of two 2-element sets. This is because a decomposition φ for which $\varphi(a)$ and $\varphi(b)$ have the same first component cannot be equivalent to a decomposition ψ for which $\psi(a)$ and $\psi(b)$ have different first components. The OMP $\mathrm{Fact}\,X$ for X a 4-element set is the shown below. It is often called MO_3.



3. RELATED CONSTRUCTIONS

The construction of $\mathrm{Fact}\,X$ for a non-empty set X can be generalized and modified in many directions. Category theory is a convenient setting to state and prove results about many of these constructions. Our aim is not to provide a version of categorical quantum mechanics, and the reader without a strong background in category theory will be able to understand the remainder of the paper without following the details of this section.

Definition 3.1. *For an object X in a category $\mathcal{C}$ that has a terminal object Ω, let* $\mathrm{Fact}\,X$ *be defined as in Definition 2.3 as the collection of all equivalence classes of binary decompositions of X.*

Here the terminal object Ω is used to define the decomposition $\iota : X \to X \times \Omega$. In general, $\mathrm{Fact}\,X$ may be a proper class. A category is called decomposition small if $\mathrm{Fact}\,X$ is a set for each object $X \in \mathcal{C}$. This is not especially important if one takes the point of view as in [19] where $\mathrm{Fact}\,X$ can be viewed as a set in some larger model of set theory.

Definition 3.2. *Let X be an object in a decomposition small category with terminal object. By* [13] *there is a well-defined partial binary operation $\oplus$ on* $\mathrm{Fact}\,X$ *given as follows. For a ternary decomposition $\gamma : X \to X_1 \times X_2 \times X_3$, then with an obvious extension* [12] *of the notation of Definition 2.3 define $[\gamma_{\{1\}\{23\}}] \oplus [\gamma_{\{3\}\{12\}}] = [\gamma_{\{13\}\{2\}}]$. Informally,*

$$[X_1 \times (X_2 \times X_3)] \oplus [X_3 \times (X_2 \times X_1)] = [(X_1 \times X_3) \times X_2]$$

Honest categories [13] are categories with finite products where projections are epic and every ternary product $X_1 \times X_2 \times X_3$ gives rise to a certain diagram that is a pushout. Disjointness [13, 16] is related to diagrams built from $X_1 \times X_2, X_1, X_2, \Omega$ being pushouts. Honesty is a general notion, but disjointness can provide problems with constructions, and Proposition 4.8, below, is not known to hold for honest categories. To deal with this problem of disjointness, strongly honest categories we introduced in [16]. They are honest categories where every n-ary product diagram is disjoint. The very strongly honest categories introduced below are an equivalent reformulation of strongly honest categories.

Definition 3.3. *A category is very strongly honest if it has finite products; projections are epic; and for every X_1, X_2, X_3, the following diagram is a pushout.*

$$\begin{array}{ccc} & X_1 \times X_2 \times X_3 & \\ \swarrow & & \searrow \\ X_1 \times X_2 & & X_2 \times X_3 \\ \searrow & & \swarrow \\ & X_2 & \end{array}$$

Theorem 3.4. *Strongly honest categories and very strongly honest categories coincide. So, for an object X in a decomposition small very strongly honest category,* $\operatorname{Fact} X$ *is an* OA.

Proof. Every strongly honest category is by definition very strongly honest. To show that a very strongly honest category is strongly honest, we must show that every binary product diagram in it is disjoint, so it must be shown that the obvious diagram built from $X_1 \times X_2, X_1, X_2, \Omega$ is a pushout. To do this, consider the ternary product $X_1 \times \Omega \times X_2$ and use the definition of very strong honesty to get that a diagram obtained from it is a pushout, and then apply basic diagram chasing. That $\operatorname{Fact} X$ is an OA for an object X in a very strongly honest category follows from the corresponding result for strongly honest categories [16]. ☐

Examples of very strongly honest categories include non-empty sets, groups, vector spaces, non-empty topological spaces, and many others where products and projections are given by cartesian products of underlying sets [13, 16]. To see why, the reader might try the simple, but illustrative, exercise of verifying that the category of non-empty sets is very strongly honest.

While honesty provides many settings in which to consider $\operatorname{Fact} X$, there are also many others. We mention one in particular, dagger biproduct categories. Their specialization to dagger compact closed categories with biproducts are the central ingredient in the categorical quantum mechanics given in [1]. The following is from [14].

Definition 3.5. *Let X be an object in a dagger biproduct category. Define* $\operatorname{Proj} X$ *to be the collection of endomorphisms p of X for which there is an endomorphism p' of X with*

(1) *p and p' are idempotent and self adjoint*
(2) $p \circ p' = 0$
(3) $p + p' = 1$

In [14] it is shown that projections correspond to biproduct decompositions $u : X \to X_1 \oplus X_2$ where u is unitary, with such u yielding the projection $p = u^\dagger \mu_1 \pi_1 u$ where μ_1, π_1 are the biproduct injection and projection to the first factor. A partial operation $\oplus$, unary operation $'$, and constants $0, 1$ are defined on $\operatorname{Proj} X$ in [14], and the following is shown.

Theorem 3.6. *For an object X in a dagger biproduct category,* $\operatorname{Proj} X$ *is an* OA.

A primary instance of this is the dagger biproduct category Hilb of Hilbert spaces and bounded linear transformations. For a Hilbert space $\mathcal{H}$ we have $\operatorname{Proj}\mathcal{H}$ is isomorphic to the OML of closed subspaces $\mathcal{C}(\mathcal{H})$ of $\mathcal{H}$.

Remark 3.7. This construction $\operatorname{Proj}\mathcal{H}$ is not the only way to realize $\mathcal{C}(\mathcal{H})$ via some type of decompositions. The direct product decompositions $\mathcal{H}$, considered as a vector space with orthogonality relation, also produces $\mathcal{C}(\mathcal{H})$. Further, considering $\mathcal{H}$ as a normed group, the idempotent endomorphisms p that satisfy $||v||^2 = ||pv||^2 + ||v - pv||^2$ produces $\mathcal{C}(\mathcal{H})$. This latter construction can be extended to produce an OMP from any normed group G with operators [10]. Here, one simply verifies directly that this is a sub-OA of the OA of all direct product decompositions of the group G. There is no need to form a category of normed groups with given type of morphism and show that such norm compatible endomorphisms p correspond to direct product decomposition in this category.

To conclude this section, we emphasize that our aim is to build a version of quantum mechanics taking an OA of decompositions of some structure X as the fundamental ingredient. We do not aim here to make a version of categorical quantum mechanics. We use category theory to establish that some construction such as $\operatorname{Fact} X$ yields an OA. The category Hilb of Hilbert spaces and bounded linear transformations does not include morphisms for position or momentum operators but this is not important. Observables in our treatment will be created from $\operatorname{Fact} X$ in an operationally motivated way, and are not simply morphisms in some ambient category. This will be treated in Section 5.

4. Group representations

Group representations in quantum mechanics provide a systematic means to exploit the symmetries inherent in a quantum system. Wigner [36], see also [31, 35], motivates group representations by considering the action of a group G on possible locations of an observer in spacetime. For example, a rotation of 3-space will move an observer from position O to O′. If s is the state of the system from the perspective of O and s' that from the perspective of O′, this rotation induces an automorphism of the convex set of states of the system.

In the standard Hilbert space approach to quantum mechanics, where a system is modeled by a Hilbert space $\mathcal{H}$, automorphisms of the state space of the system correspond to automorphisms of the OML of propositions of the system [35], and hence to automorphisms of the OML of closed subspaces of $\mathcal{H}$. Automorphisms of this OML are called physical symmetries in [35]. One can also give an argument similar to Wigner's in terms of yes/no questions asked by observers in various positions to directly motivate the connection to automorphisms of the OML of propositions of the system.

In the decompositions approach to quantum mechanics, one associates to a quantum system a structure X and an OA L built from the decompositions of X, either by $\operatorname{Fact} X$, $\operatorname{Proj} X$, or by some other means. The elements of this OA L are interpreted as the yes/no questions of the system. We extend this now by the following.

Definition 4.1. *For a quantum system with attached structure X and OA L built from the decompositions of X, we call the automorphisms of the OA L the physical symmetries of the system. The group of automorphisms of L is $\operatorname{Aut}(L)$, and a projective representation of a group G in the system represented by X is a group homomorphism*

$$\Pi : G \to \operatorname{Aut}(L)$$

While projective representations are physically motivated, a simpler notion is commonly encountered in the Hilbert space setting. A unitary representation of a group G in a Hilbert space $\mathcal{H}$ is a group homomorphism from G into the group of unitary operators of $\mathcal{H}$. We make analogous definitions in the setting of honest categories and dagger biproduct categories.

Definition 4.2. *For an object X in a category, the group of automorphisms of X is $\mathrm{Aut}(X)$. A group representation of a group G in X is a group homomorphism*

$$U : G \to \mathrm{Aut}(X)$$

We use U_g to denote the automorphism $U(g)$, and $(X, (U_g)_G)$ for the representation.

Definition 4.3. *For an object X in a dagger category, we say an automorphism u of X is unitary if $u^\dagger = u^{-1}$ and let $\mathrm{Unit}(X)$ be the group of unitary automorphisms of X. A unitary representation of a group G in X is a group homomorphism*

$$U : G \to \mathrm{Unit}(X)$$

In the Hilbert space setting, projective representations and unitary representations are related by Wigner's theorem which states that every physical symmetry of $\mathcal{H}$ is obtained from a unitary or anti-unitary operator of $\mathcal{H}$, and that two such operators induce the same physical symmetry if and only if they agree up to a scalar of unit modulus [35]. A related result holds in more general settings.

Proposition 4.4. *Let X be an object in a very strongly honest category, and Y be an object in a dagger biproduct category. Then there are group homomorphisms*

$$\Gamma : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Fact}\, X)$$
$$\Psi : \mathrm{Unit}(Y) \to \mathrm{Aut}(\mathrm{Proj}\, Y)$$

Proof. For an automorphism α of X and a binary decomposition $\varphi : X \to X_1 \times X_2$, then $\varphi \circ \alpha^{-1} : X \to X_1 \times X_2$ is also a binary decomposition of X. It is routine so show that defining $\Gamma(\alpha)([\varphi]) = [\varphi \circ \alpha^{-1}]$ gives the desired group homomorphism. For a unitary automorphism u of Y and a projection $p : Y \to Y$ in the sense of Definition 3.5, define $\Psi(u)(p) = upu^\dagger$. It is again routine to verify that Ψ is a group homomorphism. □

In general these homomorphisms Γ and Ψ are not one-one or onto. In the Hilbert space setting, the kernel of Ψ consists of the multiples of the identity operator by scalars of modulus one, and it fails to be onto because anti-unitary operators are also required in order to get all physical symmetries. Both of these failures have implications when studying representations in quantum mechanics (see [35, Ch.s 7,8]). The situation for infinite sets is much clearer as the following theorem shows [15]. It has the consequence that for infinite sets, representations correspond to projective representations.

Theorem 4.5. *For X an infinite set, $\Gamma : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Fact}\, X)$ is a group isomorphism.*

There is another way to view group representations [20, 32]. A group G can be considered as category with one object, say $*$, where each $g \in G$ is a automorphism of $*$, and with $g \circ h$ defined to be gh. For a category $\mathcal{C}$, a functor $U : G \to \mathcal{C}$ maps $*$ to some object X in $\mathcal{C}$, and takes elements $g \in G$ to automorphisms of X. Then the functors $U : G \to \mathcal{C}$ are exactly the group representations of G in objects of $\mathcal{C}$. If $U, V : G \to \mathcal{C}$ are representations with $U(*) = X$ and $V(*) = Y$, then a natural transformation $T : U \to V$ is given by a morphism $T : X \to Y$ with $T \circ U_g = V_g \circ T$ for each $g \in G$. This is the defining property of intertwining operators [23] between representations.

Definition 4.6. *For a group G and category $\mathcal{C}$, the functor category $\mathcal{C}^G$ has as its objects the functors from G to $\mathcal{C}$ and as its morphisms the natural transformations between these functors. This is called the category of G-representations in $\mathcal{C}$.*

Small modifications allow incorporation of the additional structure of dagger categories. Recall that a dagger category $\mathcal{C}$ is a category with a contravariant functor $\dagger : \mathcal{C} \to \mathcal{C}$ of period two that is the identity on objects. Each group G, considered as a 1-element category as above, naturally carries a dagger structure given by $g^\dagger = g^{-1}$.

Definition 4.7. *For a group G and dagger category $\mathcal{C}$, a representation $U : G \to \mathcal{C}$ is a unitary representation if for each $g \in G$*

$$(U_g)^\dagger = (U_g)^{-1}$$

The category $\mathcal{C}^{G^\dagger}$ of unitary representations of G in $\mathcal{C}$ is the full subcategory of $\mathcal{C}^G$ whose objects are unitary representations.

The categories $\mathcal{C}^G$ can be viewed more concretely. Their objects are functors $U : \{*\} \to \mathcal{C}$ that we represent as $(X, (U_g)_G)$ where X is the image of the object $*$ and each U_g is an automorphism of X. The morphisms in $\mathcal{C}^G$ are natural transformations between these functors, and these amount to morphisms $f : (X, (U_g)_G) \to (Y, (V_g)_G)$ where $f : X \to Y$ is a morphism in $\mathcal{C}$ with $f \circ U_g = V_g \circ f$ for each $g \in G$. Following well known results in category theory [26, p. 64], we have the following.

Proposition 4.8. *For a group G, if $\mathcal{C}$ is very strongly honest, then so is $\mathcal{C}^G$.*

Corollary 4.9. *Let X be a non-empty set, G be a group, and $(X, (U_g)_G)$ be a representation of the group G in X. Then* $\mathrm{Fact}(X, (U_g)_G)$ *is an* OA.

Remark 4.10. The above corollary could have been obtained more directly by noting that a representation $(X, (U_g)_G)$ in a set X is simply a G-set, and that this is a type of universal algebra with a family of unary operations. Then results of [10] show that $\mathrm{Fact}(X, (U_g)_G)$ is even an OMP. However, considering such things as a representation of G in a topological space X, i.e. a continuous G-set, shows the broader scope of this result.

Also following directly from well known results about functor categories is the following.

Proposition 4.11. *Let G be a group. If $\mathcal{C}$ is a dagger biproduct category, then so is $\mathcal{C}^{G^\dagger}$.*

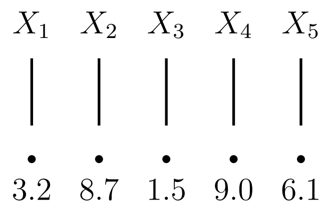
There are other results of interest in the broader context. If $\mathcal{C}$ is a monoidal category and G is a group, then $\mathcal{C}^G$ is a monoidal category with the "coordinatewise" monoidal structure. Similar comments hold for a dagger monoidal category $\mathcal{C}$ and the dagger category $\mathcal{C}^{G^\dagger}$. Further, if $\mathcal{C}$ is a strongly compact closed category with biproducts in the sense of [1], then so is $\mathcal{C}^{G^\dagger}$. So the path we take here with group representations can be implemented in the setting of [1]. A different approach to dynamics for the categorical quantum mechanics of [1] is given in [7].

5. Observables

Here we discuss observables, such as position and momentum, and especially energy, in the general setting of decompositions. From an operational viewpoint, terms such as position and momentum are abstract notions used to discuss families of compatible experiments. For example, position is a term we give to the family of measurements asking "is it here". We call this notion an observable quantity, and the particular manner in which numerical values are associated to an observable quantity, such as position, its scaling. Here we describe matters assuming that X is a structure associated to a quantum system and the corresponding OA of propositions is $\mathrm{Fact}\, X$. Obvious modifications apply if the OA of propositions were taken to be $\mathrm{Proj}\, X$ or some other. For further details about observables in this setting, see [12].

Definition 5.1. *An observable quantity is a Boolean subalgebra B of the propositions $\mathrm{Fact}\, X$ of the system.*

To discuss the matter of a scaling, assume first that we have an experiment with n outcomes. This experiment [12] corresponds to a finite Boolean subalgebra B of $\mathrm{Fact}\, X$ with n atoms, and hence to an n-ary decomposition $X \simeq X_1 \times \cdots X_n$ of X. A scaling of B is simply an assignment of numerical values to the outcomes as shown in the figure below.



This seems a suitable way to assign numerical values to outcomes in cases where we have what amounts to a discrete spectrum. For a situation such as position, where we have what amounts to a continuous spectrum, the situation is different. The notion of position at a point is treated operationally as being given by an infinite family of ever finer questions. This leads to consideration of maximally consistent families of questions for an observable quantity given by the Boolean subalgebra B of $\mathrm{Fact}\, X$. These correspond to the ultrafilters of B, that is, to the elements of the Stone space Z of B. It is to these idealized quantities that we associate numerical values for our scaling.

Definition 5.2. *A scaling of an observable quantity is a real random variable f on the Stone space Z of B, or in other words, a measurable map from Z with the Borel σ-algebra to the extended reals.*

We stress that it is the finitary decompositions that have direct physical interpretation. The passage to ultrafilters belonging to the Stone space Z of B, and then to measurable functions on Z, is made to treat directly "idealized" quantities such as position, rather than physical properties such as whether a particle is detected in a certain region. The occurrence of infinities in treating such idealized quantities is the norm, and is encountered when considering force on a classical particle governed by the inverse square law $F = GMm/r^2$ when the particle lies at the origin. We introduce terminology to reinforce this distinction between the physically realizable and idealized observables.

Definition 5.3. *Physical observables are given by an n-ary experiement corresponding to a decomposition $X \simeq X_1 \times \cdots \times X_n$ and an assignment of real numbers $\lambda_1, \ldots, \lambda_n$ to its outcomes. Idealized observables correspond to an arbitrary Boolean subalgebra B of $\mathrm{Fact}\, X$ and a scaling $f : Z \to \mathbb{R} \cup \{\pm\infty\}$ on the Stone space Z of B.*

We discuss how the standard treatment of observables in Hilbert space quantum mechanics, that given by possibly unbounded self-adjoint operators on a Hilbert space $\mathcal{H}$, fits with the description above of observable quantities and their scalings. We give a treatment that is slightly different from, but compatible with, that in [12] since it is more easily adapted to subsequent matters involving dynamics. The reader should consult [22, Chapter 5] for background.

For a self adjoint operator A, let $E^A : \mathbb{R} \to \mathrm{Proj}(\mathcal{H})$ be its associated resolution of the identity, and φ^A be its associated projection valued measure. Note that φ^A is a σ-complete OA homomorphism from the Borel subsets of the reals to $\mathrm{Proj}(\mathcal{A})$. The observable quantity one associates to A will be the complete Boolean algebra that is the image of φ^A, or more generally, the complete Boolean algebra B of projections of any abelian von Neumann algebra $\mathcal{A}$ that is affiliated with A. Here $\mathcal{A}$ is affiliated with A if $U^*AU = A$ for each unitary operator U commuting with $\mathcal{A}$ [22]. We let Z be the Stone space of B and note that since B is complete, Z is extremely disconnected, meaning that the closure of each open set in Z is open. To consider scalings of the observable quantity B, in the sense of Definition 5.2, we follow [22, Sec. 5.6].

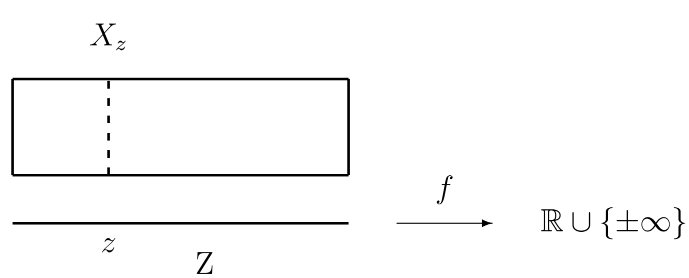
Definition 5.4. *A function $f : D \to \mathbb{C}$ that is defined and continuous on a dense open set $D \subseteq Z$ is called a normal function. A real valued normal function is called a self-adjoint function. We denote by $\mathcal{N}(Z)$ and $\mathcal{S}(Z)$ the sets of normal and self-adjoint functions on Z.*

It follows from [22, p. 344] that the self-adjoint functions on an extremely disconnected compact Hausdorff space Z correspond to the continuous functions on Z taking values in the extended reals $\mathbb{R} \cup \{\pm\infty\}$ with its usual topology. The key item is [22, Thm. 5.6.12] that we paraphrase, in part, below.

Theorem 5.5. *Let $\mathcal{A}$ be an abelian von Neumann algebra of operators on $\mathcal{H}$, B be its Boolean algebra of projections, and Z be the Stone space of B. Then there is a bijection between the self-adjoint operators A affiliated with $\mathcal{A}$ and the self-adjoint functions $\mathcal{S}(Z)$ on Z. The bounded operators belonging to $\mathcal{A}$ correspond to the continuous functions $f : Z \to \mathbb{C}$.*

This shows that self adjoint operators on $\mathcal{H}$ correspond to observable quantities B where B is a complete Boolean subalgebra of $\mathrm{Proj}(\mathcal{H})$, and the scalings of such B are not only measurable extended real valued functions on the Stone space Z of B, but are even continuous extended real valued functions on Z. The definitions of observable quantities and their scalings in Definition 5.1 and 5.2 are made more general since the indicated properties are all that is required to develop the theory. Further aspects of the standard Hilbert space treatment are compatible with our approach to observable quantities and scalings. In [12] it is shown that states on $\mathcal{H}$ give probability measures on Z, and this gives probabilities of outcomes of binary measurements of an observable when the system is in a given state, and even an expected value of an observable when the system is in a given state.

To conclude this section we comment on further ways to view matters in the setting of non-empty sets, or non-empty algebras in the sense of [5]. For X a non-empty set or algebra, the finite Boolean subalgebras with n atoms correspond to n-ary direct product decompositions $X \simeq X_1 \times \cdots \times X_n$ of X. But in these setting, the infinite Boolean subalgebras B of Fact X also have direct interpretation. Each such B gives a sheaf representation of X taken over the Stone space Z of B where the stalks X_z for $z \in Z$ are obtained as direct limits $X_z = \varinjlim_{e \in z} X_e$. For details, see [12].



It is not known whether a similar situation can be obtained, using either sheaves or related bundles, in more general settings where there are topological or analytic features.

6. Dynamics

In this section we outline the approach to dynamics in the setting of decompositions and group representations, provide a generalized time independent Schrödinger equation, and show that the standard approach to dynamics in the Hilbert space setting is an instance of this.

As a first ingredient, we need a group for time. Throughout this section we will take the additive group of real numbers for this. In Section 7 we consider also the additive group of integers when talking about physically realizable evolutions, with the view that there would be a smallest time interval that could be measured.

As motivation, consider the role played by the circle group T, i.e. the group of complex numbers of modulus 1, in the dynamics of quantum systems modeled by Hilbert spaces. For each Hilbert space $\mathcal{H}$ there is a base dynamical group $E^{\mathcal{H}}:\mathbb{R}\to\mathrm{Aut}(\mathcal{H})$ given by

$$E_t^{\mathcal{H}}\,v \;=\; e^{-it}\,v$$

We refer to $E^{\mathcal{H}}$ as the natural frequency of $\mathcal{H}$. Note that each decomposition of $\mathcal{H}$ into a sum of orthogonal closed subspaces is compatible with the natural frequency since $v=v_1+\cdots+v_n$ implies that $e^{-it}\,v=e^{-it}\,v_1+\cdots+e^{-it}\,v_n$.

Definition 6.1. *For a category $\mathcal{C}$, call $\hat{\mathcal{C}}=\mathcal{C}^{\mathbb{R}}$ an assignment of natural frequencies in $\mathcal{C}$, and for a dagger category $\mathcal{C}$, call $\hat{\mathcal{C}}=\mathcal{C}^{\mathbb{R}^\dagger}$ an assignment of unitary natural frequencies in $\mathcal{C}$.*

In categories $\hat{\mathcal{C}}$, objects can be thought of as objects of $\mathcal{C}$ equipped with an "internal clock". One may find it more natural to consider full subcategories of $\hat{\mathcal{C}}$ consisting of one member of $\hat{\mathcal{C}}$ for each object X of $\mathcal{C}$, so that as in the Hilbert space case each object in $\mathcal{C}$ is given one choice of natural frequency, but this is not vital. If one were to prefer such subcategories, they should be chosen to be closed under the pertinent structure of $\hat{\mathcal{C}}$ such as finite products or biproducts. The results of Section 4 provide the following.

Theorem 6.2. *If $\mathcal{C}$ is a very strongly honest category, biproduct category, or dagger biproduct category, then for any object $(X,(E_t)_{\mathbb{R}})$ in $\hat{\mathcal{C}}$, its direct product decompositions* $\mathrm{Fact}(X,(E_t)_{\mathbb{R}})$ *form an* OA.

Suppose that H is the Hamiltonian for a quantum system modeled by the Hilbert space $\mathcal{H}$, and assume that H has finitely many eigenvalues $\lambda_1,\ldots,\lambda_n$. Then there is a dynamical group $U:\mathbb{R}\to\mathrm{Aut}\ \mathcal{H}$ induced by this Hamiltonian H as follows. If $v\in\mathcal{H}$ has eigenspace decomposition $v=v_1+\cdots+v_n$ then

$$U_t\,v \;=\; e^{-i\lambda_1 t}\,v_1+\cdots+e^{-i\lambda_n t}\,v_n \tag{6.1}$$

This may be rewritten as $U_t=e^{-iHt}$, a generalized form of the Schrödinger equation.

For an object $\hat{X}=(X,(E_t)_{\mathbb{R}})$ in $\hat{\mathcal{C}}$ we may consider its decompositions in the category $\hat{\mathcal{C}}$. These are decompositions $\hat{X}\simeq\hat{X}_1\times\cdots\times\hat{X}_n$, where $\hat{X}_i=(X_i,(E_t^i)_{\mathbb{R}})$, that are decompositions of the underlying structure X and are compatible the assigned natural frequencies. In situations described in Section 4, the decompositions of $\hat{X}$ again form an OA or OMP $\mathrm{Fact}\,\hat{X}$ or $\mathrm{Proj}\,\hat{X}$, and we may then speak of its observable quantities and scalings as before. In the following we assume we are dealing with $\mathrm{Fact}\,\hat{X}$, but the situation for $\mathrm{Proj}\,\hat{X}$ is similar.

Definition 6.3. *For an object $\hat{X}$ in $\hat{\mathcal{C}}$, let H be a physical observable of* $\operatorname{Fact}\hat{X}$ *associated with the finite decomposition $\hat{X} \simeq \hat{X}_1 \times \cdots \times \hat{X}_n$ and the scaling $\lambda_1, \ldots, \lambda_n$. Then define the representation $E^H : \mathbb{R} \to \operatorname{Aut}\hat{X}$ by*

$$E_t^H\, x \;=\; (E^1_{\lambda_1 t}\, x_1, \ldots, E^n_{\lambda_n t}\, x_n)$$

We have a familiar situation. An observable quantity H of $\hat{X}$, which might be called the Hamiltonian of the system, determines the dynamical group $U : \mathbb{R} \to \operatorname{Aut}\ \hat{X}$ of the system. We write this as follows, a type of generalized time independent Schrödinger's equation.

$$U_t \;=\; E_t^H \tag{6.2}$$

Remark 6.4. In the generalized Schrödinger's equation (6.2), an element x is broken into components, and the components at higher energy levels λ evolve at more rapid rates. This can be viewed as an extension of the wave formalism where frequency is related to energy, that is, components at higher energy levels evolve more rapidly.

The construction of representations of $\mathbb{R}$ in Definition 6.3 applies to physical observables, that is, to finite decompositions and their scalings. For an idealized observable quantity H we might aim to produce a representation E^H of $\mathbb{R}$ in $\operatorname{Fact} X$ as the result of limiting process of representations produced from physical observables that are approximations to H. Such a limiting process seems to require additional topological or analytic structure on X and/or $\operatorname{Fact} X$. Such structure is not available in the general setting, but does exist in a number of cases of interest, such as normed groups [3, 10], vector bundles [17], and perhaps also in the structures given in [12, Sec. 6.4], and others. While analysis of these particular situations has not yet been made, we can consider matters in the standard Hilbert space setting where there is ample topological and analytic structure on $\mathcal{H}$ and $\operatorname{Proj}\mathcal{H}$. Here idealized observables correspond to self-adjoint operators H on $\mathcal{H}$ with infinite spectrum. We begin with an analysis of bounded self-adjoint operators.

Proposition 6.5. *Let H be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then for any $n \in \mathbb{N}$, there is a physical observable H_n on $\mathcal{H}$ so that for any $t \in [-n, n\,]$*

$$||\, e^{-itH} - E_t^{H_n}\, || < 1/n$$

Thus, there is a sequence H_n $(n \in \mathbb{N})$ of physical observables so that for any $v \in \mathcal{H}$ and $t \in \mathbb{R}$ we have that $E_t^{H_n} v \to e^{-itH} v$ in the topology of $\mathcal{H}$.

Proof. Let $\mathcal{A}$ be the abelian von Neumann algebra generated by H and the identity I. Let B be the complete Boolean algebra of projections of $\mathcal{A}$ and let Z be the Stone space of B. There there is an isomorphism Γ from $\mathcal{A}$ to $C(Z)$, the continuous complex-valued functions on Z. Let $f : Z \to \mathbb{R}$ be the continuous function corresponding to the self-adjoint operator H.

Given $n \in \mathbb{N}$, there is a finitely valued function $f_n : Z \to \mathbb{R}$ such that $||\, f - f_n\, || < 1/n^2$ where this norm is the usual sup norm on $C(Z)$. Suppose that the finitely valued function f_n takes the values $\lambda_1, \ldots, \lambda_{m_n}$. Then there are clopen subsets $Z_1, \ldots, Z_{m_n}$ of Z where f_n is the constant λ_i on Z_i. For each $1 \le i \le m_n$, let P_i be the projection in $\mathcal{A}$ corresponding to the clopen set Z_i, and set

$$H_n = \lambda_1 P_1 + \cdots + \lambda_{m_n} P_{m_n}$$

Since H_n is a self-adjoint operator with a finite spectrum, it corresponds to a physical observable of $\mathcal{H}$. Further, the function $f_n \in C(Z)$ corresponds to the element H_n of $\mathcal{A}$ via Γ. Since Γ preserves norms, then $||\, H - H_n\,|| = ||\, f - f_n\,|| < 1/n^2$. Since the functional calculus on $\mathcal{A}$ transfers to $C(Z)$ we have that for any time t, that Γ takes e^{-itH} to e^{-itf} and $E_t^{H_n} = e^{-itH_n}$ to e^{-itf_n}. So $||\, e^{-itH} - E_t^{H_n}\,|| = ||\, e^{-itf} - e^{itf_n}\,||$.

For any real numbers x, y, we have $|\, e^{ix} - e^{iy}\,| < |\, x - y\,|$. It follows that for $t \in [-n, n\,]$,

$$\begin{aligned} ||\, e^{-itf} - e^{itf_n}\,|| &= \sup\{\,|\, e^{-itf(z)} - e^{-itf_n(z)}\,| : z \in Z\} \\ &\leq \sup\{\,|\, tf(z) - tf_n(z)\,| : z \in Z\} \\ &= \sup\{\,|\, t\,|\,|\, f(z) - f_n(z)\,| : z \in Z\} \\ &\leq n||\, f - f_n\,|| \\ &\leq 1/n \end{aligned}$$

This shows that for any $t \in [-n, n\,]$, that $||\, e^{-itH} - E_t^{H_n}\,|| < 1/n$. Suppose $v \in \mathcal{H}$ and $t \in \mathbb{R}$. Then for any $\epsilon > 0$ we can find N so that for any $n > N$ we have $1/n\,||\, v\,|| < \epsilon$ and $t \in [-n, n\,]$. For any $n > N$ we have $||\, e^{-itH} - E_t^{H_n}\,|| < 1/n$, and hence $||\, e^{-itH}v - E_t^{H_n}v\,|| < 1/n\,||\, v\,|| < \epsilon$. So the sequence $E_t^{H_n}v$ converges to $e^{-itH}v$. □

It remains to consider the situation for an ideal observable given by an unbounded self-adjoint operator H. Again, the aim is to approximate the unitary representation e^{-itH} given by H in the standard approach to quantum mechanics by a family of representations given by physical observables.

Theorem 6.6. *Let H be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there is a sequence H_n $(n \in \mathbb{N})$ of physical observables so that for any $v \in \mathcal{H}$ and $t \in \mathbb{R}$ we have that $E_t^{H_n}v \to e^{-itH}v$ in the topology of $\mathcal{H}$.*

Proof. Suppose H is affiliated with the abelian von Neumann algebra $\mathcal{A}$. Let B be the complete Boolean algebra of projections of $\mathcal{A}$, let Z be the Stone space of B, and let $f : D \to \mathbb{R}$ be the self-adjoint function associated to H, defined on the dense open set $D \subseteq Z$. Stone duality provides a bijection between elements of B and clopen subsets of Z, and for a projection $P \in B$ we let $\beta(P)$ be the clopen subset of B corresponding to P.

Let $\mathfrak{X}$ be the collection of all sets S of pairwise disjoint subsets of D with each $A \in S$ clopen in Z. Then $\mathfrak{X}$ is non-empty and closed under unions of chains. By Zorn's lemma, $\mathfrak{X}$ contains a maximal member M. Since M is a collection of pairwise disjoint clopen subsets of Z, there is a family Q_i $(i \in I)$ of pairwise orthogonal projections in B with $M = \{\beta(Q_i) : i \in I\}$. These pairwise orthogonal projections correspond to pairwise orthogonal subspaces of $\mathcal{H}$, and since $\mathcal{H}$ is separable, this family Q_i $(i \in I)$ must be countable. We may assume that it is indexed as Q_n for $n \in \mathbb{N}$. We show that $\bigcup_{\mathbb{N}} \beta(Q_n)$ is dense in Z. Indeed, if A is a non-empty open set in Z, then since D is dense open in Z and Z has a basis of clopen sets, there is a non-empty clopen set K that is contained in $D \cap A$. Then by the maximality of M, it cannot be the case that K is disjoint from Q_n for each $n \in \mathbb{N}$. Since $\bigcup_{\mathbb{N}} \beta(Q_n)$ is dense in Z, it follows that $\bigvee_{\mathbb{N}} Q_n = 1$. For each $n \in \mathbb{N}$ define $P_n = \bigvee\{Q_k : k \leq n\}$. Then the family P_n $(n \in \mathbb{N})$ is an increasing chain of projections in B with $\bigvee_{\mathbb{N}} P_n = 1$ and $\beta(P_n) \subseteq D$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, the operator P_nH is affiliated with $\mathcal{A}$, and its corresponding self-adjoint function is $f_n : Z \to \mathbb{R}$ given by

$$f_n(z) = \begin{cases} f(z) & \text{if } z \in \beta(P_n) \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

Then for each $n \in \mathbb{N}$, P_nH is a bounded-self adjoint operator. So we may apply Proposition 6.5 to obtain a physical observable H_n so that for any $t \in [-n, n\,]$ we have

$$||\, e^{-itP_nH} - E_t^{H_n}\, || < 1/n \tag{6.4}$$

Suppose we are given time $t \in \mathbb{R}$ and $v \in \mathcal{H}$. Since $\bigvee_{\mathbb{N}} P_n = 1$, we have $\lim P_n v = v$. So for any $\epsilon > 0$ there is an $N_0 \in \mathbb{N}$ so that for all $n > N_0$ we have $||\, P_n^{\perp} v\, || < \epsilon$ where $P_n^{\perp}$ is the orthogonal projection to P_n. With the given $t \in \mathbb{R}$ and $v \in \mathcal{H}$, let $\epsilon > 0$. Choose $N \in \mathbb{N}$ large enough so that for each $n > N$ we have $t \in [-n, n\,]$, $||\, P_n^{\perp} v\, || < \epsilon/4$, and $n > 2\, ||\, v\, ||/\epsilon$. Note

$$||\, e^{-itH} v - E_t^{H_n} v\, || \;\leq\; ||\, e^{-itH} v - e^{-itP_nH} v\, || \;+\; ||\, e^{-itP_nH} v - E_t^{H_n} v\, || \tag{6.5}$$

Writing $v = P_n v + P_n^{\perp} v$ we have

$$||\, e^{-itH} v - e^{-itP_nH} v\, || \;\leq\; ||\, e^{-itH} P_n v - e^{-itP_nH} P_n v\, || \;+\; ||\, e^{-itH} P_n^{\perp} v - e^{-itP_nH} P_n^{\perp} v\, || \tag{6.6}$$

The continuous complex valued functions on Z corresponding to e^{-itH} and e^{-itP_nH} both agree with e^{-if} on the clopen set $\beta(P_n)$. Thus $e^{-itH} P_n$ and $e^{-itP_nH} P_n$ are equal. So the first term in the right side of inequality (6.6) is equal to zero. We then obtain from (6.6) that

$$||\, e^{-itH} v - e^{-itP_nH} v\, || \;\leq\; ||\, e^{-itH} - e^{-itP_nH}\, || \cdot ||\, P_n^{\perp} v\, || \tag{6.7}$$

Both e^{-itH} and e^{-itP_nH} are unitary, hence $||\, e^{-itH} - e^{-itP_nH}\, || \leq ||\, e^{-itH}\, || + ||\, e^{-itP_nH}\, || \leq 2$. Then for $n > N$, from our assumption that $||\, P_n^{\perp} v\, || < \epsilon/4$ we have

$$||\, e^{-itH} v - e^{-itP_nH} v\, || \;\leq\; \epsilon/2 \tag{6.8}$$

For the second term in inequality (6.5) we have

$$||\, e^{-itP_nH} v - E_t^{H_n} v\, || \;\leq\; ||\, e^{-itP_nH} - E_t^{H_n}\, || \cdot ||\, v\, || \tag{6.9}$$

Since $n > N$ we have $t \in [-n, n\,]$, and our choice of physical observables H_n was made so that $||\, e^{-itP_nH} - E_t^{H_n}\, || < 1/n$ for all $t \in [-n, n\,]$. Also, since $n > N$ we have that $n > 2\, ||\, v\, ||/\epsilon$. Thus

$$||\, e^{-itP_nH} v - E_t^{H_n} v\, || \;\leq \epsilon/2 \tag{6.10}$$

So for $n > N$ we have $||\, e^{-itH} v - E_t^{H_n} v\, || \leq \epsilon$. So $E_t^{H_n} v \to e^{-itH} v$ in the topology of $\mathcal{H}$. □

Remark 6.7. This settles the case for the base dynamical group $E^{\mathcal{H}} : \mathbb{R} \to \mathrm{Aut}(\mathcal{H})$ given by $E^{\mathcal{H}}_t v = e^{-it}v$ that is used in the standard approach to quantum mechanics. In the above proofs we use many properties not only of Hilbert spaces, but also of this base dynamical group. Even the modest move to considering which other base dynamical groups over $\mathcal{H}$ might give a reasonable dynamics poses challenges.

7. Assignments of natural frequencies in sets and topological spaces

Here we consider assignments of natural frequencies to sets, and later to topological spaces. As mentioned in Section 6, this involves choosing a group G for time, and then looking at representations of G in $\mathrm{Aut}(X)$ for a set X. In terms of Definition 6.1, such representations are objects in the category SET^G where SET is the category of non-empty sets. We first make some general comments about representations of groups in sets. Here we can profitably view matters from the perspective of universal algebra. See [25, p. 108].

Proposition 7.1. *For a group G, an object of SET^G is a G-set $(X, (U_g)_G)$ that consists of a non-empty set X with unary operations U_g for $g \in G$ with $U_g \circ U_h = U_{gh}$ for all $g, h \in G$, and a morphism of SET^G is a homomorphism of G-sets. Thus SET^G is the variety of G-sets.*

To understand the nature of G-sets, the notion of an orbit in a G-set is of basic use.

Definition 7.2. *For a G-set $(X, (U_g)_G)$, the orbit of an element $x \in X$ is $\{U_g(x) : g \in G\}$. A G-set is called transitive if it has a single orbit.*

It is easily seen that an orbit of a G-set naturally forms a G-set, that the union of any family of disjoint G-sets naturally forms a G-set, and that each G-set is the union of its orbits. Thus, to understand the nature of G-sets, for a given G, it suffices to understand the transitive G-sets. The following is well known, see for example [25, Thm. 3.4].

Proposition 7.3. *For any subgroup of a group G, the collection G/H of left cosets of H in G is a G-set with the group action $U_g(xH) = gxH$. Further, each transitive G-set is isomorphic to G/H where H is the stabilizer subgroup of any element of the G-set.*

We now apply these considerations to dynamics. The situation is simplest when the group chosen for time is the additive group of integers $\mathbb{Z}$. This choice of group is physically motivated by the view that there is a smallest time interval, or at least a smallest time interval that can in practice be measured. Here, the description of $\mathbb{Z}$-sets is obvious from Proposition 7.3.

Proposition 7.4. *The transitive $\mathbb{Z}$-sets are isomorphic to the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}$ with the obvious action of $\mathbb{Z}$.*

Describing the transitive G-sets for G being the additive group of rationals or reals requires a notion familiar from universal algebra [5, 25].

Definition 7.5. *An algebra A is subdirectly irreducible if for each embedding $\varphi : A \to \prod_I A_i$ of A into a product of algebras with $\pi_i \circ \varphi$ onto for each projection π_i, there is some index i with $\pi_i \circ \varphi$ an isomorphism.*

Theorem 7.6. *For a group G, the subdirectly irreducible transitive G-sets are exactly those G-sets isomorphic to the set of cosets G/H, with obvious action, where H is a completely meet irreducible subgroup of G. Here a completely meet irreducible subgroup is one that cannot be expressed as the intersection of any family of subgroups of G that properly contain it.*

Proof. Suppose that X is a transitive G-set. Then Proposition 7.3 shows that X is isomorphic to G/H for some subgroup H of G. Then [25, Lemma 4.20] shows that the congruence lattice of the G-set G/H is isomorphic to the interval $[H, G]$ of the subgroup lattice of G. Then the general characterization of subdirectly irreducible algebras [25, Thm. 4.40] shows that G/H is subdirectly irreducible iff H is completely meet irreducible. □

As with any universal algebra [5], any G-set is given by a subdirect product of the family of subdirectly irreducible G-sets that are homomorphic images of it. If G is a tranisitve G-set, then its homomorphic images are also transitive. So a description of the subdirectly irreducible transitive G-sets goes a considerable way to describing all transitive G-sets, and hence all G-sets. In general, describing the subdirectly irreducible transitive G-sets will be a difficult problem, but when G is a divisible abelian group, such as $\mathbb{Q}$ or $\mathbb{R}$, things are more tractable. We recall that for a prime p, the Prüfer p^∞ group is the subgroup of $\{z \in \mathbb{C} : |z| = 1\}$ consisting of all p^n*-th* roots of unity for some natural number n.

Proposition 7.7. *Let G be a divisible group. Then the subdirectly irreducible transitive G-sets are isomorphic to G/H with the natural action where G/H is isomorphic to a Prüfer p^∞ group for some prime p.*

Proof. By Theorem 7.6 the subdirectly irreducible transitive G-sets are isomorphic to G/H where H is a completely meet irreducible subgroup of G. Since every subgroup H of an abelian group is normal, each G/H is an abelian group. The conditions of H being completely meet irreducible is equivalent to the group G/H being subdirectly irreducible. A quotient of a divisible group is divisible, and it is known [8, p. 576] that the subdirectly irreducible divisible groups are exactly the Prüfer groups p^∞ for some prime p. □

It is tempting to see a resemblance between these representations using Prüfer p^∞ groups and the familiar representation of $\mathbb{R}$ in the complex line given by circle group. But there is an important aspect lacking, that of continuity. Indeed, when considering representations of a group G in the automorphisms of a set X, we lack the analytic structure on X to even consider notions of continuity. We extend our setting by allowing topological structure on X and considering topological groups G.

Definition 7.8. *Let G be a topological group and let X be a topological space. Consider a group representation $(X, (U_g)_G)$ as a map $U : G \times X \to X$ where $U(g, x) = U_g(x)$. We say that this representation is continuous if U is a continuous function.*

For a continuous transitive representation $(X, (U_g)_G)$ of a topological group G, we have that $(X, (U_g)_G)$ is isomorphic as a G-set to G/H where H is the stabilizer of an element $x_0 \in X$. Since $H = U(\,\cdot\,, x_0)^{-1}(x_0)$, it is a closed subgroup of G. The following result [18, p. 120] shows that each closed subgroup H of G gives rise to a continuous transitive representation of G.

Proposition 7.9. *If G is a topological group and H is a closed subgroup of G, then G/H is Hausdorff in the quotient topology and the natural transitive coset representation G/H is a continuous representation of G.*

Under suitable restrictions, the isomorphism of transitive group representations given in Proposition 7.3 will extend to a homeomorphism of the underlying spaces. The following result is given in [18, Thm. 3.2].

Theorem 7.10. *Let G be a locally compact group with a countable base for its topology and let X be a locally compact Hausdorff space. Then for any continuous, transitive representation $(X, (U_g)_G)$ there is a closed subgroup H of G and a bijection*

$$f : G/H \to X$$

that is both an isomorphism of G-sets and a homeomorphism of spaces.

We shift attention to the primary matter at hand, continuous representations of $\mathbb{R}$.

Proposition 7.11. *Let $R^+ = \{\lambda \in \mathbb{R} : \lambda \geq 0\}$. The closed subgroups of $\mathbb{R}$ are the following.*

(1) $H_0 = \mathbb{R}$
(2) $H_\lambda = \{n\lambda : n \in \mathbb{Z}\}$ *for* $\lambda \in \mathbb{R}^+ \setminus \{0\}$
(3) $H_\infty = \{0\}$

Proof. Clearly each of these is a closed subgroup of $\mathbb{R}$. Conversely, suppose that H is a closed subgroup of $\mathbb{R}$. If H is not equal to $\{0\}$, then $H^* = H \cap \{x : x > 0\}$ is non-empty and bounded below, so has an infinum λ. If $\lambda \notin H^*$, then it is easy to see that for any $\epsilon > 0$ that there are elements $x, y \in H$ with $|x - y| < \epsilon$. It follows that H is dense in $\mathbb{R}$, and as it is closed, that $H = \mathbb{R}$. If $\lambda \in H^*$, then it follows that $H = H_\lambda$. □

Definition 7.12. *For $\lambda \in \mathbb{R}^+ \cup \{\infty\}$, let $(T_\lambda, (V_t)_\mathbb{R})$ be the continuous coset representation of $\mathbb{R}$ given by $\mathbb{R}/H_\lambda$ with the quotient topology. Then*

(1) T_0 *is a singleton*
(2) T_λ *for* $\lambda \in \mathbb{R}^+ \setminus \{0\}$ *is a unit circle with the action* V_t *being rotation by* $2\pi t/\lambda$
(3) T_∞ *is the reals with the action* V_t *being translation by* t

For each $\lambda \in \mathbb{R}^+ \setminus \{0\}$, T_λ bears similarity to the usual circle group. As topological spaces, they are homeomorphic. However as $\mathbb{R}$-sets, they are not. As $\mathbb{R}$-sets, we have that T_λ is isomorphic to $T_{\lambda'}$ iff $\lambda = \lambda'$. As a consequence of Theorem 7.10 and Proposition 7.11, we have the following.

Corollary 7.13. *Each continuous, transitive representation of $\mathbb{R}$ in a locally compact Hausdorff space X is isomorphic as an $\mathbb{R}$-set, and homeomorphic, to T_λ for some $\lambda \in \mathbb{R}^+ \cup \{\infty\}$.*

This result cannot be extended to general topological spaces since any $\mathbb{R}$-set with the indiscrete topology will provide a continuous representation of $\mathbb{R}$. However, it would be desirable to have the result for general Hausdorff spaces without the assumption of local compactness. We do not know a complete answer, but a partial result is provided by the following.

Proposition 7.14. *Suppose that X is a topological space and that $(X, (U_t)_\mathbb{R})$ is a continuous representation of $\mathbb{R}$. Let $\lambda \in \mathbb{R}^+ \cup \{\infty\}$ be such that $(X, (U_t)_\mathbb{R})$ is isomorphic to T_λ as an $\mathbb{R}$-set. Then there is a continuous $\mathbb{R}$-set isomorphism $f : T_\lambda \to X$.*

Proof. Consider the case T_∞, which is the reals with $V_t(s) = s + t$. Choose some $x_0 \in X$, and define $f : T_0 \to X$ by setting $f(s) = U_s(x_0)$. This is seen to be an isomorphism of $\mathbb{R}$-sets. Since T_0 is a sequential space, to show that f is continuous it suffices to show that if $s_n \to s$ in T_0, then $f(s_n) \to f(s)$. But $f(s_n) = U_{s_n}(x_0)$ and $f(s) = U_s(x_0)$. Since $U : \mathbb{R} \times X \to X$ is continuous and $s_n \to s$ in $\mathbb{R}$, we have $U(s_n, x_0) \to U(s, x_0)$, and hence $U_{s_n}(x_0) \to U_s(x_0)$. So $f(s_n) \to f(s)$.

The case for T_λ, where $\lambda \in \mathbb{R}^+$ is similar. Elements of T_λ are equivalence classes s/H_λ. Define $f(s/H_\lambda) = U_s(x_0)$. For a sequence $s_n/H_\lambda \to s/H_\lambda$ where $s \notin H_\lambda$ we may choose the s_n

and s to belong to $(0,\lambda)$ with $s_n \to s$ in $\mathbb{R}$, and the argument is as above. If $s \in H_\lambda$, apply the homeomorphism $V_{\lambda/2}$ to the sequence. Then $f \circ V_{\lambda/2}(s_n/H_\lambda) \to f \circ V_{\lambda/2}(s/H_\lambda)$, and as $f \circ V_{\lambda/2} = U_{\lambda/2} \circ f$ and $U_{\lambda/2}$ is a homeomorphism, we have $f(s_n/H_\lambda) \to f(s/H_\lambda)$. □

Corollary 7.15. *Suppose that X is a Hausdorff space and that $(X,(U_t)_\mathbb{R})$ be a continuous transitive representation of $\mathbb{R}$. Then this representation is isomorphic and homeomorphic to T_λ for some $\lambda \in \mathbb{R}^+$, or is isomorphic as an $\mathbb{R}$-set to the reals, and the topology on X is a weaker Hausdorff topology than the usual topology on $\mathbb{R}$.*

Proof. The spaces T_λ for $\lambda \in \mathbb{R}^+$ are compact, and a continuous bijection from a compact Hausdorff space to a Hausdorff space is a homeomorphism. If $(X,(U_t)_\mathbb{R})$ is isomorphic as an $\mathbb{R}$-set to the reals T_∞, then there is a continuous $\mathbb{R}$-set isomorphism $f : T_\infty \to X$. Treating this isomorphism as the identity, the topology on X is weaker than the usual topology on $\mathbb{R}$. □

The following gives an example, applicable in the standard Hilbert space setting, of a transitive $\mathbb{R}$-set that is algebraically isomorphic to the reals, but has a strictly weaker topology.

Example 7.16. Let $\mathbb{C}^2$ be the complex plane with the standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$. Let P_i for $i = 1,2$ be the projection onto the span of e_i. Then for real numbers α_1, α_2 consider the self-adjoint operator $H = \alpha_1 P_1 + \alpha_2 P_2$. There is a representation $U_t = e^{-itH}$ of the reals in $\mathbb{C}^2$ given by this self-adjoint operator. For $v = (\beta_1, \beta_2)$ we have

$$U_t(v) = (e^{-i\alpha_1 t}\beta_1, e^{-i\alpha_2 t}\beta_2)$$

If $\alpha_i = 0$, each point in the image of P_i is in a singleton orbit of type T_0 and each other point has an orbit type the same as the points in the image of P_j. Suppose $\alpha_i \neq 0$ for $i = 1,2$. Then each point in the image of P_i has an orbit of type $T_{2\pi/\alpha_i}$. For a point v that is not in the image of either P_i, the type of its orbit depends on α_1/α_2. If this is a rational p/q in lowest terms, then the orbit of v is of type T_λ where $\lambda = 2\pi p/\alpha_1 = 2\pi q/\alpha_2$. If α_1/α_2 is irrational, then each v not in the image of P_1 or P_2 has orbit of type T_∞. These irrational orbits have a subspace topology that is weaker than the standard topology on the reals since open sets pick up points on future traversals.

8. Concluding remarks

Our aim is to provide a path to including a dynamics in the decompositions approach to quantum mechanics. The idea is to attach to each object an internal clock E, that is, a representation of the reals in the automorphism group of the object. One then considers decompositions of the object that are compatible with this internal clock. A dynamics for the system is obtained from an energy observable H for the system based on the simple idea that factors of the system at higher energy levels have their internal clocks run more rapidly. This produces a type a generalized time independent Schrödinger equation for the system,

$$U_t = E_t^H$$

Our approach is shown to be consistent with the standard Hilbert space approach to dynamics. For a self-adjoint operator H with finite spectrum, the dynamics associated to H agrees exactly with ours. When the spectrum of H is infinite, even unbounded, the dynamics given by the standard e^{-itH} of Hilbert space quantum mechanics is the limiting process of the dynamics via decompositions given by finitary approximations to H.

First steps were made towards classifying internal clocks in the setting of sets and topological spaces. These are representations of the reals in the permutation group of a set, or continuous representations of the reals in the group of homeomorphisms of a space. In each case similarities were seen with the representations via the circle group familiar from the standard Hilbert space setting.

The program outlined here gives the first steps towards incorporating group representations into the decompositions approach to quantum mechanics. With an eye to the future, basics of general group representations are developed in the decompositions setting. It is shown that many of the categorical techniques that are used in the decompositions approach, such as honest and dagger biproduct categories, are compatible with the implementation of representations. While the decompositions approach is not intended as a sort of categorical quantum mechanics along the lines of [1], the use of such categorical tools is very helpful in its development.

There remain many directions for study. There are a number of settings somewhat close to the Hilbert space setting that allow the analytic tools needed to treat idealized observables. These include normed groups with operators [10], vector bundles [17], and the normed sets that were considered in [12]. Further development of representations of the reals including versions of a Stone-von Neumann theorem for 1-parameter groups in these setting would be of interest. Even the standard Hilbert space setting may present open ground if a base representation other than the usual one were taken. In the study of representations of more general groups, a likely starting spot might be the finite crystallographic groups considered in [33].

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References

[1] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 415-425, IEEE Comput. Soc. Press, Los Alamitos, CA, 2004.

[2] S. Abramsky and B. Coecke, *Categorical quantum mechanics*, in: Handbook of quantum logic and quantum structures, Elsevier, (2009), 26–324.

[3] N. H. Bingham and A. J. Ostaszewski, *Normed versus topological groups: dichotomy and duality*, Dissertationes Math. (Rozprawy Mat.) 472 (2010), 138 pp.

[4] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Ann. of Math. (2) 37 (1936), no. 4, 823-843.

[5] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics (78), Springer, 1981.

[6] M. D. Chiara, R. Giuntini, and R. J. Greechie, *Reasoning in Quantum Theory Sharp and Unsharp Quantum Logics*, Trends in Logic, Vol. 22, Kluwer, Dordrecht/Boston/London, 2004.

[7] S. Gogioso, *Categorical quantum mechanics for Schrödinger's equation*, arXiv:1501.06489v2.

[8] P. A. Grillet, *Abstract Algebra*, Graduate Texts in Mathematics, Springer, 2007.

[9] T. Hannan and J. Harding, *Automorphisms of decompositions*, to appear in Math. Slovaka.

[10] J. Harding, *Decompositions in quantum logic*, Trans. Amer. Math. Soc. 348 (1996), no. 5, 1839-1862.

[11] J. Harding, *Regularity in quantum logic*, Internat. J. of Theoret. Phys. 37 (1998), no. 4, 1173-1212.

[12] J. Harding, *Axioms of an experimental system*, Internat. J. of Theoret. Phys. 38 (1999), no. 6, 1643-1675.

[13] J. Harding, *Orthomodularity of decompositions in a categorical setting*, Internat. J. of Theoret. Phys. 45 (2006), no. 6, 1117-1127.

[14] J. Harding, *A link between quantum logic and categorical quantum mechanics*, Internat. J. Theoret. Phys. 48 (2009), no. 3, 769-802.

[15] J. Harding, *Wigner's theorem for an infinite set*, arXiv:1604.06973.

[16] J. Harding and Taewon Yang, *Sections in orthomodular structures of decompositions*, to appear in The Houston J. Math.

[17] J. Harding and Taewon Yang, *The logic of bundles*, to appear in Internat. J. of Theoret. Phys.

[18] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, Elsevier, 1978.

[19] H. Herrlich and G. Strecker, *Category Theory; An Introduction*, Allyn and Bacon series in advanced mathematics, Allyn and Bacon, Boston, 1973.

[20] A. Joyal and R. Street, *An introduction to Tanaka duality and quantum groups*, Lecture Notes in Math. 1488, Springer-Verlag, Berlin, 1991, 411-492.

[21] G. Kalmbach, *Orthomodular Lattices*, London Mathematical Society Monographs, 18. Academic Press, Inc. London, 1983.

[22] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. 1: Elementary Theory*, Academic Press, New York, 1983.

[23] G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics*, A lecture-note volume by W. A. Benjamin, Inc., New York-Amsterdam, 1963.

[24] S. Mac Lane, *Categories for the Working Mathematician*, second ed., Springer, New York, 1998.

[25] R. McKenzie, G. McNulty, and W. Taylor, *Algebras, Lattices, Varieties,* Vol. I, Wadsworth & Brooks/Cole, Monterey, 1987.

[26] B. Mitchell, *Theory of Categories*, Academic Press, New York, 1965.

[27] D. Kh. Mushtari, *Projection logics in Banach spaces*, Soviet Math. (Iz. VUZ) 33 (1989), 59-70.

[28] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Fundamental Theories of Physics, 44. Kluwer Academic Publishers Group, Dordrecht, 1991.

[29] E. Prugovecki, *Quantum Mechanics in Hilbert Space*, second ed., Academic Press, New York, 1981.

[30] P. Selinger, *Dagger compact closed categories and completely positive maps: (extended abstract)*, Electronic Notes in Theoretical Computer Science, Volume 170, March 2007, Pages 139-163.

[31] S. Frank Singer, *Linearity, Symmetry, and Prediction in the Hydrogen Atom*, Springer, 2005.

[32] R. Street, *Monoidal categories for the combinatorics of group representations*, electronically available notes.

[33] M. Tinkham, *Group Theory and Quantum Mechanics*, Dover, 2003.

[34] U. Uhlhorn, *Representation of symmetry transformations in quantum mechanics*, Ark. Fys. 23 (1963), 307-340.

[35] V. S. Varadarajan, *Geometry of Quantum Theory*, Second Ed., Springer-Verlag, 1985.

[36] E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of the Atomic Spectra*, Academic Press, 1959.

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