# Modularity is not canonical

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**Abstract.** In [2] it was shown that the canonical extension of a bounded modular lattice need not be modular. The proof was indirect, using a deep result of Kaplansky. In this note we give an explicit example.

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## 1. Introduction

In [2] it was shown that the canonical extension of a modular lattice need not be modular. The proof was through a result of Kaplansky [3], that a complete modular ortholattice is a continuous geometry, and von Neumann's work on dimension functions of continuous geometries. Together, these provide that a complete modular ortholattice cannot have an infinite pairwise perspective set. Therefore, the modular ortholattice of finite and cofinite dimensional subspaces of an infinite dimensional Hilbert space cannot be embedded into a complete modular ortholattice. Since the canonical completion of an ortholattice is an ortholattice, this implies that the canonical completion of this modular lattice of finite and cofinite dimensional subspaces cannot be modular.

I have been asked on several occasions for a more explicit reason for the failure of modularity to be canonical. Such is provided in the following. This example grew from discussions with Christian Herrmann related to the old problem of whether the variety of modular ortholattices is generated by ones derived from vector spaces.

## 2. Setup

For $L$ a bounded lattice, its canonical extension $L^\sigma$ is characterized [1] up to isomorphism as the unique complete lattice that contains $L$ as a sublattice where $L \leq L^\sigma$ is dense and compact. Density means that each element of $L^\sigma$ is a join of meets of elements of $L$ and a meet of joins of elements of $L$.

Compactness means that for $S, T$ subsets of $L$, if $\bigwedge S \leq \bigvee T$ then there are finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$. The canonical extension satisfies [1, Lemma 3.2] the complete distributive law when applied to a meet of up-directed joins of elements of $L$ and to a join of down-directed meets of elements of $L$.

**Lemma 2.1.** *Suppose $A, B_n, C_n$ for $n \in \mathbb{N}$ are elements of $L$ with the following properties.*

(1) *The $B_n$ are decreasing.*
(2) *The $C_n$ are increasing.*
(3) *$B_n \nleq A$ for each $n$.*
(4) *$A \wedge B_n \neq 0$ for each $n$.*
(5) *$C_n \neq 1$ for each $n$.*
(6) *$B_{n+1} \wedge C_n = 0$ for each $n$.*
(7) *$(A \wedge B_n) \vee C_n = 1$ for each $n$.*

*Then $F = \bigwedge B_n$, $G = \bigwedge (A \wedge B_n)$ and $I = \bigvee C_n$ together with $0, 1$ form a pentagon in $L^\sigma$.*

*Proof.* Surely $G \leq F$. Compactness and (3) give $F \nleq A$. Then as $G \leq A$, we have $G < F$. Compactness and (4), (5) give $G \neq 0$ and $I \neq 1$. Having all $B_n = 1$ or all $C_n = 0$ contradicts assumptions (3), (6), and (7), thus $F \neq 1$ and $I \neq 0$. It remains to show that $F \wedge I = 0$ and $G \vee I = 1$. Note that $F \wedge I = B_1 \wedge B_2 \wedge \cdots \wedge \bigvee C_n$. By complete distributivity applied to a meet of up-directed joins from $L$ we have

$$F \wedge I = \bigvee_n \bigwedge_m (B_m \wedge C_n)$$

So (6) gives $F \wedge I = 0$. That $I \vee G = 1$ follows similarly using complete distributivity applied to a join of down-directed meets from $L$ and (7). □

## 3. The example

Let $K$ be a field of characteristic not equal 2. For $n \in \mathbb{N}$ let $V_n$ be the vector space $K^n$ and $S_n$ be the lattice of subspaces of $V_n$. For each $n \in \mathbb{N}$ there is a lattice embedding $\sigma_n : S_n \to S_{2n}$ taking a subspace $A$ to the set of vectors $(x_1, \ldots, x_{2n})$ where both $(x_1, \ldots, x_n)$ and $(x_{2n+1}, \ldots, x_{2n})$ belong to $A$. For $n < m$ composition gives a map $\varphi_{n,m} : S_{2^n} \to S_{2^m}$. A vector in $V_{2^m}$ can be broken into pieces of length $2^n$. The map $\varphi_{n,m}$ takes $A$ to the set of vectors where each such piece belongs to $A$.

**Definition 3.1.** Let $L = \lim S_{2^n}$ be the direct limit.

Since $L$ is the limit of a family of complemented modular lattices, $L$ is a complemented modular lattice. If $K$ is chosen to be a subfield of the complex numbers that is closed under conjugation, and $S_n$ is endowed with the orthocomplementation given by the canonical scalar product, then $L$ would be even an orthocomplemented modular lattice.

Elements of $L$ are equivalence classes, but we treat $A \in S_{2^n}$ as an element of $L$ meaning its equivalence class. If $A \in S_{2^n}$ and $B \in S_{2^m}$ where $n < m$ then the meet of their equivalence classes is formed by taking the equivalence class of the meet of $\varphi_{n,m}(A)$ and $B$. We abuse notation and denote this $A \wedge B$. Similar comments apply to join.

**Definition 3.2.** Let $A \in S_2$, and for each $n \in \mathbb{N}$ let $B_n \in S_{2^{n+1}}$ and $C_n \in S_{2^{n+2}}$ be as follows.

$$\begin{aligned} A &= < x_1, 0 > \\ B_n &= < x_1, x_2, x_1, x_2, \ldots, x_1, x_2 > \\ C_n &= < x_1, x_1, x_3, x_4, \ldots, x_{2^{n+2}-1}, 2x_{2^{n+2}-1} > \end{aligned}$$

So $A$ is all vectors in $V_2$ whose second entry is 0, $B_n$ is all vectors in $V_{2^{n+1}}$ whose odd entries are all equal and whose even entries are all equal, and $C_n$ is all vectors in $V_{2^{n+2}}$ whose first two entries are equal and whose last entry is twice its second last. As noted, we consider these as elements of $L$.

**Proposition 3.3.** *The elements $A$, $B_n, C_n$ for $n \in \mathbb{N}$ satisfy the conditions of Lemma* 2.1*, so the canonical extension $L^\sigma$ is not modular.*

*Proof.* As mentioned, computations are done by moving the given subspaces to the appropriate $S_n$. Considering $B_n$ in $S_{2^{n+2}}$ it is all vectors in $V_{2^{n+2}}$ where the first and second halves have the properties that their odd components are equal and their even components are equal. Thus $B_n$ contains $B_{n+1}$, so the $B_n$ are decreasing. In a similar way, the $C_n$ are seen to be increasing. Considered in $S_{2^{n+1}}$ we have $A$ is all vectors in $2^{n+1}$ whose even components are 0. So for each $n$, $B_n \nleq A$ and $A \wedge B_n \neq 0$. Clearly $C_n \neq 1$ for each $n$. This provides the first five conditions.

For the final two conditions, define the normalized dimension of $A \in S_n$ to be $\mathrm{Dim}\, A = \dim A/n$. It is easily seen that normalized dimension satisfies the familiar dimension formula $\mathrm{Dim}\, P + \mathrm{Dim}\, Q = \mathrm{Dim}(P \vee Q) - \mathrm{Dim}(P \wedge Q)$, and the maps $\varphi_{n,m}$ preserve normalized dimension. Note $\mathrm{Dim}\, B_n = 2/2^{n+1}$, $\mathrm{Dim}(A \wedge B_n) = 1/2^{n+1}$ and $\mathrm{Dim}\, C_n = (2^{n+2} - 2)/(2^{n+2})$. It follows that $\mathrm{Dim}(A \wedge B_n) = \mathrm{Dim}\, B_{n+1} = 1 - \mathrm{Dim}\, C_n$. To show the last two conditions of Lemma 2.1, and additionally that $C_n$ is a common complement of $A \wedge B_n$ and $B_{n+1}$, it is enough to show $B_{n+1} \wedge C_n = 0$ and $(A \wedge B_n) \wedge C_n = 0$.

Now $B_{n+1}$ is all vectors of length $2^{n+2}$ whose odd entries agree and whose even entries agree, and $C_n$ is all vectors of length $2^{n+2}$ whose first two entries agree and whose last entry is twice the second last entry. So $B_{n+1} \wedge C_n = 0$. Also $A \wedge B_n$ is all vectors of length $2^{n+2}$ whose even entries are all zero, whose odd entries in the first half agree, and whose odd entries in the second half agree. So for any vector in $(A \wedge B_n) \wedge C_n$, the condition on $C_n$ that the first two entries agree forces all entries in the first half of the vector to be zero, and the condition on $C_n$ that the last entry is twice the second last forces all entries in the second half to be zero. So $(A \wedge B_n) \wedge C_n = 0$. □

## References

[1] Gehrke, M., Harding, J.: Bounded lattice expansions. J. of Algebra. **238**, 345–371 (2001)

[2] Harding, J.: Canonical completions of lattices and ortholattices. Tatra Mt. Math. Publ. **15**, 85–96 (1998)

[3] Kaplansky, I.: On orthocomplemented complete modular lattice is a continuous geometry. Ann. of Math. **61**, 524–157541 (1955)

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