Daggers, kernels, Baer *-semigroups and orthomodularity

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Overview

We discuss ways to create orthomodular structures (OMLs, OMPs) from objects in various types of categories.

In large part, this is a survey of old results (30's - 70's), as well as newer ones, adapted to this categorical setting.

A few small, but useful observations are thrown in.

Basic idea

Theorem Let $E(R)$ be the idempotents of a ring R with unit. Set $e \leq f$ iff $ef = e = fe$ and $e' = 1 - e$. Then $E(R)$ forms an OMP.

Note This is a non-commutative version of the familiar connection between Boolean rings and Boolean algebras.

Idea Try to apply this to the endomorphisms of an object A in a category. We can get by with (much) less than a ring, so this becomes fairly general.

The start — von Neumann regular rings

Defn R is regular if each x has y with $xyx = x$, which occurs iff each principal right ideal is generated by an idempotent.

Defn R is Rickart if for each x the annihilator $\{z : xz = 0\}$ is gen'd by an idempotent; Baer if this holds for annihilators of subsets.

Note Regular $\Rightarrow$ Rickart.
Note Annihilators are linked to the categorical notion of kernels.

Von Neumann regular rings

Theorem For R regular, equivalence classes of idempotents $E(R)$ under the quasiorder $e \sqsubseteq f \Leftrightarrow fe = e$ form a complemented modular lattice. This lattice is complete iff R is Baer.

Note The quasiorder $\sqsubseteq$ is different than the partial order $\leq$.

Note Not all complemented modular lattices arise this way, many do, and from a unique such regular ring. Coordinatization.

*-Rings

Defn A *-ring is one with an involution *, meaning an operation satisfying $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, and $aa^* = 0 \Rightarrow a = 0$.

Defn A projection is an idempotent e with $e = e^*$.

Note Involutions * are similar to daggers † for a category.

Note For e, f projections $ef = e \Rightarrow e^* = (ef)^* \Rightarrow e = fe$.

Note For projections $\sqsubseteq$ and $\leq$ agree.

Note The involution * provides a fragment of commutativity!

*-Rings

Theorem The projections of a regular *-ring form a MOL. If the ring is Baer, this is a complete MOL so a continuous geometry.

Theorem The projections of a Rickart *-ring form an OML.

Note The orthocomplement of e is not e^*, but $1 - e$.

Note In each case, the MOL or OML is a sub-OMP of $E(R)$.

The role of *

Modular case If M is coordinatized by a regular ring R, each orthocomplementation on M produces an involution on R. So the existence of * can be proved, and is tied to underlying geometry.

Orthomodular case Here there is no uniqueness of coordinatization and no direct link between orthocomplementations and involutions. The practical role of * is to provide a fragment of commutativity.

Lets drop $+$ and move away from rings ...

Foulis semigroups (Foulis 1960)

Defn A Foulis semigroup is a semigroup S with zero, unit, an involution $*$ and unary $'$ where x' is the unique projection with the annihilator $\{x : ax = 0\}$ equal to $x'S$.

Example The multiplicative fragment of a Rickart *-ring.

Theorem The projections x' of a Foulis semigroup form an OML and each OML arises this way.

Note To coordinatize an OML by a Foulis semigroup construct S from its Galois connections. This isn't unique and doesn't reduce to the geometric coordinatization for MOLs.

Doing without *

For MOLs it can't be done, you get one whether you want it or not. For OMLs it can, but its not pretty. The idea is to replace * with the bits of commutativity it provides.

Theorem For a semigroup S with zero, unit and unary operation $'$, let $S' = \{a' : a \in S\}$. Suppose

1. Each a' is idempotent and $a'S = \{x : ax = 0\}$
2. For $e, f \in S'$, $ef = fe \Leftrightarrow e'f = fe' \Leftrightarrow ef = (ef)''$.

Then S' is an OML.

Lets move to a categorical setting ...

Dagger kernel categories

Defn A dagger kernel category is a dagger category with zero object where each f has a kernel k with $k^\dagger k = 1$.

Note Generalize this to a weak dagger kernel category.

- Replace $k^\dagger k = 1$ with $kk^\dagger m = m$ for all m with $fm = 0$.
- Don't require a zero object, just zero maps between objects.

Examples The category of finite dimensional Hilbert spaces with linear maps and the category of OMLs with Galois connections as maps are dagger kernel categories. Any Foulis semigroup is a one-element weak dagger kernel category.

Dagger kernel categories

Theorem (Crown 1975)

Each object in a weak dagger kernel category yields an OML.

Proof. $EndA$ is a semigroup with zero, unit, and involution. If $f \in EndA$ has weak dagger kernel k, the projection $kk^\dagger$ generates the annihilator $\{m : fm = 0\}$. So $EndA$ is a Foulis semigroup.

Note Heunen and Jacobs rediscovered this and added many new connections to categorical logic.

Note Crown worked a bit more generally with daggers and kernels to produce OMPs from each object. His main interest was categories of vector bundles which are not dagger kernel categories.

Partial semigroups

Setting S is a set with a partial binary operation.

Say S is commutative if one side of $xy = yx$ being defined implies both sides are defined and the two are equal, etc.

Defn S is a partial semigroup if it is associative.

Note Crown used creatures called partial Baer-* semigroups to make OMPs from objects in a category that is nearly a dagger kernel category, such as a category of vector bundles.

But things can be easier ...

Orthomodular partial semigroups

Defn An orthomodular partial semigroup (OPS) is a commutative, associative, idempotent, partial semigroup with a zero, unit and unary operation $'$ satisfying

1. $xx' = 0$
2. xy defined implies $x'y$ defined.
3. $xy = xz$ and $x'y = x'z$ imply $y = z$ (jointly monic).

Example Take the idempotents $E(R)$ of a ring and restrict multiplication to commuting elements. Set $x' = 1 - x$.

Note These are the bits of ring structure needed to make an OMP!

OPSs and OMPs

Theorem Suppose S is an OPS and P is an OMP. Then

1. $\Psi S = (S, \leq, ', 0, 1)$ is an OMP where $e \leq f$ iff $ef = e = fe$.
2. $\Phi P = (P, \cdot)$ is an OPS where $ef = e \wedge f$ for e, f compatible.
3. $\Phi\Psi S = S$.
4. $\Psi\Phi P = P$.

Note One can make this into a full categorical isomorphism.

Note This is nearly trivial, but worthwhile knowing!

Remarks

All the above constructions amount to instances of OPSs.

If you want to build an OMP from idempotent endomorphisms of an object in a category, you want to build an OPS inside of *EndA*. A dagger is not vital, but helps in getting bits of commutativity.

This all builds from the basic example of the OMP $E(R)$.

Special cases

Prop If $\mathcal{C}$ is semiadditive, the idempotents $e \in EndA$ that have e' with $e + e' = 1$ and $ee' = 0 = e'e$ form an OPS hence an OMP.

Prop If $\mathcal{C}$ is semiadditive and has a dagger, same as above if we also require $e^\dagger = e$.

Note If $\mathcal{C}$ has (dagger) biproducts, it is semiadditive, so OMPs.

Examples

FDHilb We have dagger kernels and dagger biproducts. Both yield the OML of closed subspaces of H.

Rel We have dagger kernels and dagger biproducts. Both yield the power set of X.

OML Objects are OMLs and morphisms are Galois connections. This has dagger kernels and dagger biproducts. The dagger kernels of L yield L itself, dagger biproducts of L yield the center of L.

Examples

Mat_K This has daggers, kernels, and dagger biproducts, but not dagger kernels. Yields OMPs.

$VectBund(X)$ This is a dagger biproduct category, but does not have kernels. Yields OMPs.

Note All these were considered by Foulis or Crown in some form.

Note All but *OML* are strongly compact closed, *OML*???

Last comment

There is another way to construct orthostructures from objects.

Theorem If $\mathcal{C}$ has finite products, projections are epic, and each

$$\begin{array}{ccccc} & & A \times B \times C & & \\ & \swarrow & & \searrow & \\ A \times B & & & & B \times C \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

is a pushout

Then the direct product decomp's of an object form an OA.

Note This seems fundamentally different from other techniques.

Thank you for listening.

Papers at www.math.nmsu.edu/∼JohnHarding.html