Products or Sums?

John Harding

New Mexico State University
www.math.nmsu.edu/~JohnHarding.html

jharding@nmsu.edu

Amsterdam, May 2015

Foreword

I'd like to describe an ongoing project.

Some portions are well developed and settled.

I'd also like to discuss future directions and obstacles. I would be happy to have suggestions.

The role of projection operators

In the standard Hilbert space formulation of QM, projections play a central role. Our key ingredients.

$\mathcal{Q}$ = the orthomodular lattice of projections of $\mathcal{H}$

$\mathcal{S}$ = the convex set of states

$\mathcal{O}$ = the observables

$\mathcal{B}$ = the Borel algebra of $\mathbb{R}$

$\mathcal{G}$ = a Lie group

The role of projection operators

The Spectral Theorem

Observables correspond to σ-homomorphisms $E : \mathcal{B} \to \mathcal{Q}$

Gleason's Theorem

States correspond to σ-additive $s : \mathcal{Q} \to [0, 1]$

Wigner's Theorem

Unitary and anti-unitary maps of $\mathcal{H}$ correspond to auto's of $\mathcal{Q}$

There are a number of areas where Wigner's theorem is pertinent.

Wigner's Theorem

The dynamical group of the system

This is a continuous group homomorphism $\mathrm{U} : \mathbb{R} \to \mathrm{Aut}(\mathcal{H})$. The unitary U_t is the time evolution operator from t_0 to t.

Stone's Theorem

The dynamical groups are exactly given by $\mathrm{U}_t = e^{i\mathrm{H}t}$ for some $\mathrm{H} \in \mathcal{O}$ called the Hamiltonian or energy operator. This may be viewed as an abstract form of Schrödinger's equation.

Wigner's Theorem

Group Representations

A continuous homomorphism $\prod : \mathcal{G} \to \mathrm{Aut}(\mathcal{H})$.

These include dynamical groups, the finite symmetry groups one considers for crystals, SO(3) from considerations of shapes of orbitals, the Lorentz group for free particles, and so forth.

Wigner's theorem relates this to $\prod : \mathcal{G} \to \mathrm{Aut}(\mathcal{Q})$.

Note: Do not forget the topology — on $\mathcal{G}$ and $\mathcal{H}$ and $\mathcal{Q}$.

Wigner's Theorem

The Fundamental Theorem of Projective Geometry

The subspaces of a vector space $\mathcal{V}$ form a projective geometry $\mathcal{P}$ and the auto's of $\mathcal{P}$ correspond to skew-automorphisms of $\mathcal{V}$.

Wigner's theorem is a direct analog. This underscores the geometric view of quantum theory.

Program

- Try to replace $\mathcal{H}$ with another structure X.
- To build an omp $\mathcal{Q}$ from X.
- To use this as a basis of developing aspects of QM.

How to build $\mathcal{Q}$ and from what?

Rather than view projections as closed subspaces or summands, we view them as corresponding to direct products $\mathcal{H} \simeq \mathcal{H}_1 \times \mathcal{H}_2$.

Theorem One can build an omp $\mathcal{Q}$ from the direct product decompositions of virtually any type of structure X where

- the orthocomplement of $X \simeq A \times B$ is $X \simeq B \times A$
- $X \simeq A \times (B \times C) \ \leq \ X \simeq (A \times B) \times C$.

Note A superposition is not $u + v$, but rather (u, v).

Examples

Some examples of the structures X to which this theorem applies are the following.

- sets
- sets with valuation $v : X \to [0, \infty)$
- G-sets
- groups, rings
- normed groups
- graphs
- topological spaces
- vector bundles (with inner product)
- An abstract object in a suitable type of category

Constructing $\mathcal{Q}$ for these examples

For normed groups and vector bundles, it is similar to the Hilbert space setting.

For groups $\mathcal{Q}$ is built as complementary pairs of its modular lattice of normal subgroups. Rings and modules are similar.

For sets, G-sets, sets with valuation, $\mathcal{Q}$ is constructed from ordered pairs of equivalence relations.

Note In some cases such as vector bundles and normed groups, $\mathcal{Q}$ forms a topological omp in the sense of Wilce.

The Spectral Theorem

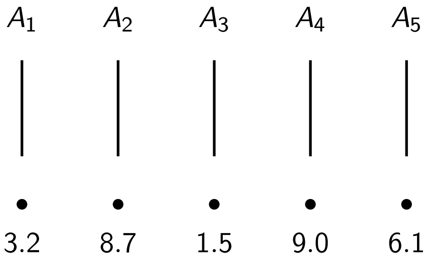
For $\mathcal{Q}$ constructed from X, there is plausible physical reasoning for construction of observables.

- Call n-ary decompositions $X \simeq A_1 \times \cdots \times X_n$ n-ary experiments.
- Members of $\mathcal{Q}$ are binary experiments or questions.
- Finite Boolean $\mathcal{B} \leq \mathcal{Q}$ correspond to n-ary experiments.
- Arbitrary $\mathcal{B} \leq \mathcal{Q}$ correspond to sheaf rep's of X in good cases.

The Spectral Theorem

A Boolean subalgebra of $\mathcal{Q}$ consists of compatible questions that can be asked simultaneously such as "is it here" or "here". We call a Boolean subalgebra of $\mathcal{Q}$ an observable quantity.

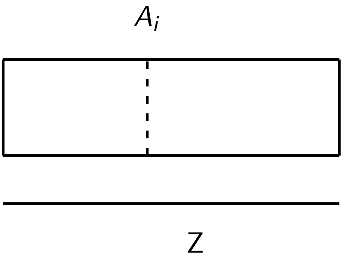
To assign numbers to on observable quantity we give a numerical value to each outcome. We call this a scaling.



The Spectral Theorem

Let $\mathcal{B}$ be an infinite Boolean subalgebra with Stone space Z.

A scaling of $\mathcal{B}$ is a measurable map $\varphi : \mathrm{Z} \to \mathbb{R}$.



An observable is an observable quantity together with its scaling. We obtain a calculus of compatible observables working in $C(\mathrm{Z})$ with $A + B$, A^2, e^A as in the Hilbert space setting.

States

A state is a (σ) additive map $s : \mathcal{Q} \to [0,1]$

States give probabilities for questions in the usual way. We also obtain other properties as in the Hilbert space setting.

Theorem Suppose $\mathcal{B}$ is an observable quantity with Stone space Z. Then each state s gives a measure μ_s on Z. Further, the observable consisting of $\mathcal{B}$ scaled by φ has expected value in state s

$$\int \varphi d\mu_s$$

States

The previous result made use of states on $\mathcal{Q}$. Existence?

Theorem If X is a set with valuation, normed group, or vector bundle, then $\mathcal{Q}$ has a full set of states.

Note For any infinite set X, there are no states on $\mathcal{Q}$. There are however an abundance of "partial states".

Note No attempt has yet been made at Gleason's Theorem in this setting, except for vector bundles. This seems difficult.

Wigner's Theorem

For $\mathcal{Q}$ built from a vector space V, elements of $\mathcal{Q}$ are ordered pairs (S, T) of complementary subspaces of V. Ovchinnikov gave a version of Wigner's Theorem for such $\mathcal{Q}$.

Theorem Automorphisms of $\mathcal{Q}$ are pairs (α, β) where α is an auto and β a dual auto of the subspace lattice of V.

Note The fundamental theorem of projective geometry describes such α and β in terms of skew automorphisms of V.

Note Similar results hold for groups, rings, and modules but we don't have the description of α and β.

Wigner's Theorem

I'd like a version of Wigner's Theorem for $\mathcal{Q}$ built from a set X.

Conjecture Aut $\mathcal{Q}$ is isomorphic to Aut X.

When $|X|$ has one or two prime factors, or equals 8, the conjecture is false. The only case I know it is true is when $|X| = 27$.

When $|X| = 27$ $\mathcal{Q}$ is huge, and the proof is many pages of difficult combinatorial arguments. It shows a lot about $\mathcal{Q}$ such as it being highly transitive. This is open for X infinite.

Group representations

Definition A representation of G is a group homomorphism

$$\Pi : G \to \mathrm{Aut}\ X$$

If our objects X lie in some category $\mathcal{C}$, then a representation of G is a functor from the 1-element category G to $\mathcal{C}$.

The representations of G in $\mathcal{C}$ are a category $\mathcal{C}^G$ with morphisms natural transformations. Products of rep's, etc have meaning.

Group representations

Classically, irreducible representations give physically quantities. Irreducible representations of S0(3) give shapes of orbitals, etc.

The correct notion in the more general setting seems to be that of subdirectly irreducible representations.

Theorem Representations of G in Set are G-sets. The subdirectly irreducible representations are the G-sets G/H where H is a meet irreducible subgroup.

It would be of interest to compare set representations of finite symmetry groups to the usual linear representations.

Dynamical groups

A basic feature of the standard Hilbert space approach is the unitary representation

$$\mathrm{E} : \mathbb{R} \to \mathrm{Aut}\ \mathbb{C} \qquad \text{where} \qquad \mathrm{E}_t(z) = e^{it} z$$

If the energy observable H has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$

Then the dynamical group U of the system has

$$\mathrm{U}_t v = e^{3it} v_1 + e^{5it} v_2$$

where $v = v_1 + v_2$ is the eigenstate superposition for v.

Dynamical groups

This suggests the following approach to the general situation.

Consider a category $\mathcal{C}$ such as sets, and the category $\mathcal{C}^{\mathbb{R}}$ of sets with a base dynamical group. An object in $\mathcal{C}^{\mathbb{R}}$ is

$$\mathsf{E}^A : \mathbb{R} \to \operatorname{Aut} A$$

Then take the decompositions $\mathcal{Q}$ of an object in this category. These will be the decompositions that are compatible with the group actions.

Dynamical groups

Now suppose that some observable we call the Hamiltonian has the following decomposition and scaling.

$$\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ \Big| & \Big| & \Big| & \Big| & \Big| \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 3 & 8 & 1 & 9 & 6 \end{array}$$

Then the dynamical operator U of the system takes has

$$\mathrm{U}_t(a_1, \ldots, a_5) = \left(\mathrm{E}^{A_1}_{3t}\,(a_1), \;\ldots\;, \mathrm{E}^{A_5}_{6t}\,(a_5) \right)$$

Tensor products

For systems with structures X_1, X_2 and logics $\mathcal{Q}_1, \mathcal{Q}_2$ we want a structure X for the compound system so that for its logic $\mathcal{Q}$:

1. There is $f : \mathcal{Q}_1 \times \mathcal{Q}_2 \to \mathcal{Q}$
2. This f preserves orthogonal joins in each argument
3. For states σ_i of $\mathcal{Q}_i$, there is a state ω of $\mathcal{Q}$ with

$$\omega(f(q_1, q_2)) = \sigma_1(q_1)\sigma_2(q_2)$$

Tensor products

Incorporating tensor products with the rest of the program remains a problem area. In some cases close to Hilbert space it seems that progress can be made with modest compromises

- vector bundles with inner product
- normed abelian groups (torsion free)

Another path is via categorical QM ...

Tensor products

Let $\mathcal{C}$ be a dagger symmetric monoidal category with biproducts.

Theorem For an object $A \in \mathcal{C}$

1. The biproduct decompositions $\mathcal{Q}(A)$ form an orthoalgebra.
2. $\mathcal{Q}(A \otimes B)$ has many properties of a tensor product of logics.

Note This likely has many extensions (no dagger, products)

Note This fits with the scheme $\mathcal{C}^{\mathbb{R}}$ for dynamics and $\mathcal{C}^{G}$ for rep's.

Tensor products

The problem in merging decompositions with the categorical approach is the scalars.

The decomposition approach fundamentally uses the reals in its scalings, measures, and dynamics.

While the categorical approach has its scalars $\mathcal{C}(I, I)$, I can't connect to this in the ways needed.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding