Proximity Frames

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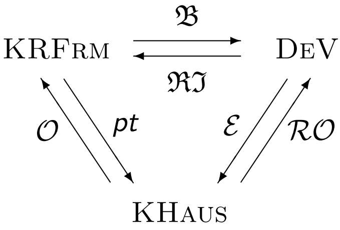
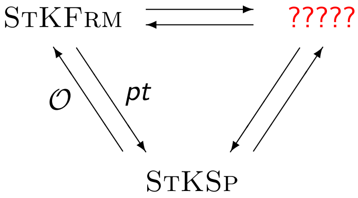
Overview

There is a triangle of equivalences, and dual equivalences, involving the category KHaus of compact Hausdorff spaces.



Overview

Stably compact spaces generalize compact Hausdorff spaces, and it is known that the above triangle of equivalences can be partially extended as follows.



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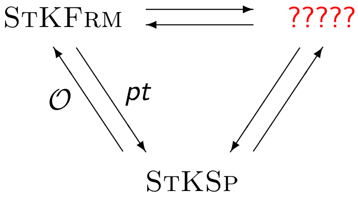


Our aim is to complete this picture.

Overview

A very incomplete history

Alexandroff → Smirnoff

Freudenthal → de Groot → de Vries

Isbell

Compendium and its authors

Banaschewski

Smyth

Jung and Sünderhoff

I. Basic dualities

In a frame, we use $a \ll b$ for the way below relation, and $a \prec b$ for the well inside relation $(\neg a \vee b = 1)$.

Definition A frame is compact regular if

1. $1 \ll 1$.
2. $a = \bigvee \{b : a \prec b\}$.

$\mathrm{K}\textsc{RFrm}$ is the category of compact regular frames and frame homomorphisms.

I. Basic dualities

Consider the frame of opens of a compact Hausdorff space.

1. $A \prec B$ means $\mathrm{C}A \subseteq B$.
2. $B = \bigcup\{A : \mathrm{CA} \subseteq B\}$ for each B.
3. This frame is compact regular.
4. In this case $\prec$ and $\ll$ agree, but not in general.

I. Basic dualities

Definition A de Vries algebra is a complete Boolean algebra D with relation $\prec$ (proximity) that satisfies

1. $1 \prec 1$.
2. $a \prec b$ implies $a \leq b$.
3. $a \leq b \prec c \leq d$ implies $a \prec d$.
4. $a \prec b, c$ implies $a \prec b \wedge c$.
5. $a \prec b$ implies $\neg b \prec \neg a$.
6. $a \prec b$ implies there exists c such that $a \prec c \prec b$.
7. $a \neq 0$ implies there exists $b \neq 0$ such that $b \prec a$.

Definition A de Vries morphism is a map f that preserves bounds and finite meets, and satisfies

1. $a \prec b$ implies $\neg f(\neg a) \prec f(b)$.
2. $f(a) = \bigvee\{f(b) : b \prec a\}$.

Definition Composition of de Vries morphisms is given by

$$(g \star f)(a) = \bigvee\{gf(b) : b \prec a\}.$$

Proposition de Vries algebras and morphism form a category DeV.

I. Basic dualities

Definition For a subset A of a de Vries algebra $\mathcal{D}$, set

$$\Uparrow A = \{b : a \prec b \text{ for some } a \in A\}.$$

$$\Downarrow A = \{b : b \prec a \text{ for some } a \in A\}.$$

A filter F is round if $F = \Uparrow F$, and an ideal I is round if $I = \Downarrow I$.

Definition The ends $\mathcal{E}\mathcal{D}$ are the maximal round filters. They are topologized by having all ϕa as a basis where

$$\phi a = \{E : a \in E\}.$$

I. Basic dualities

There is a dual equivalence $\text{KHAUS} \underset{pt}{\overset{\mathcal{O}}{\rightleftarrows}} \text{KRFRM}$

$\mathcal{O}X$ is the frame of opens of X, and $\mathcal{O}f = f^{-1}$.

$pt\ L$ is the space of points of the frame L and $pt\ f = - \circ f$.

I. Basic dualities

There is a dual equivalence $$\mathrm{KH{\small AUS}} \underset{\mathcal{E}}{\overset{\mathcal{RO}}{\rightleftarrows}} \mathrm{D{\small E}V}$$

$\mathcal{RO}X$ is the complete Boolean algebra of regular open sets of X with $A \prec B$ iff $\mathrm{C}A \subseteq B$. On maps, $\mathcal{RO}f = \mathrm{IC}f^{-1}$.

$\mathcal{ED}$ is the space of ends of $\mathcal{D}$ and $\mathcal{E}f = \twoheaduparrow f^{-1}$.

I. Basic dualities

There is an equivalence $\textsc{DeV} \underset{\mathcal{B}}{\overset{\mathcal{RI}}{\rightleftarrows}} \textsc{KRFrm}$

$\mathcal{RI}\,\mathcal{D}$ is the frame of round ideals of $\mathcal{D}$ and $\mathcal{RI}f = \twoheaddownarrow f$.

$\mathcal{B}$ is the Booleanization functor, where $\mathcal{B}L = \{\neg\neg a : a \in L\}$ with $\prec$ the restriction of well inside. On maps $\mathcal{B}f = \neg\neg f$.

II. Stably compact spaces

Definition A space X is stably compact if it is

1. Compact.
2. Locally compact.
3. Sober.
4. The intersection of (two) compact saturated sets is compact.

Saturated means being the intersection of opens.

Definition A continuous map is proper if the preimage of a compact saturated set is compact saturated.

II. Stably compact spaces

A stably compact space has three associated topologies:

τ	given topology
τ^k	having the compact saturated sets as its closed sets
π	the topology generated by τ and τ^k

Call τ^k the co-compact topology, π the patch topology.

II. Stably compact spaces

Definition A compact frame is stably compact if

1. $a = \bigvee\{b : b \ll a\}$.
2. $a \ll b, c$ implies $a \ll b \wedge c$.

Definition A frame homomorphism is proper if $a \ll b \Rightarrow fa \ll fb$.

II. Stably compact spaces

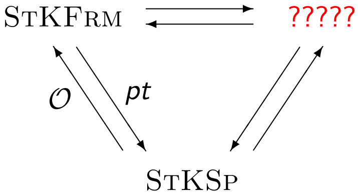
Theorem $\mathcal{O}$ and *pt* restrict to a dual equivalence

$$\textsc{StKSp} \underset{pt}{\overset{\mathcal{O}}{\rightleftarrows}} \textsc{StKFrm}$$

$\textsc{StKSp}$ = stably compact spaces and proper continuous maps,
$\textsc{StKFrm}$ = stably compact frames and proper frame homo's.

III. Proximity frames

We begin new work, that of filling in the question marks.



III. Proximity frames

Definition A proximity frame is a frame L with a binary relation $\prec$ that satisifes

1. $0 \prec 0$ and $1 \prec 1$.
2. $a \prec b$ implies $a \leq b$.
3. $a \leq b \prec c \leq d$ implies $a \prec d$.
4. $a, b \prec c$ implies $a \vee b \prec c$.
5. $a \prec b, c$ implies $a \prec b \wedge c$.
6. $a \prec b$ implies there exists c with $a \prec c \prec b$.
7. $a = \bigvee \{b : b \prec a\}$.

III. Proximity frames

Definition A proximity morphism is a map f that preserves bounds and finite meets and satisfies

1. $a_1 \prec b_1$ and $a_2 \prec b_2$ imply $f(a_1 \vee a_2) \prec f(b_1) \vee f(b_2)$.
2. $f(a) = \bigvee\{f(b) : b \prec a\}$.

Definition Composition of proximity morphisms is given by

$$(g \star f)(a) = \bigvee\{gf(b) : b \prec a\}.$$

Definition $\textsc{PrFrm}$ = proximity frames and their morphisms.

Examples of proximity frames

1. Any de Vries algebra.
2. Any frame with $\leq$ as its proximity.
3. A strong inclusion $\triangleleft$ on a frame.
4. Any stably compact frame with $\ll$ as its proximity.
5. The ideal lattice of a distributive lattice with $\ll$ as proximity.
6. On $Pow\,\mathbb{N}$ define $A \prec B$ if either $\begin{cases} A \subseteq B \text{ and } A \text{ finite} \\ B = \mathbb{N} \end{cases}$

Note A proximity frame whose underlying frame is Boolean need not be de Vries. Our proximities forget $\neg$. This is a good thing.

IV. First equivalences

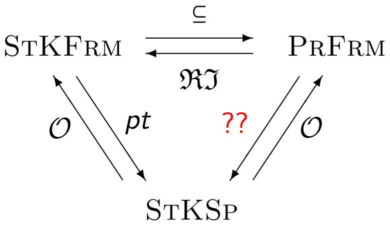
Recall, a stably compact frame is a proximity frame under $\ll$.

Theorem There is an equivalence

$$\textsc{StKFrm} \underset{\mathcal{RI}}{\overset{\subseteq}{\rightleftarrows}} \textsc{PrFrm}$$

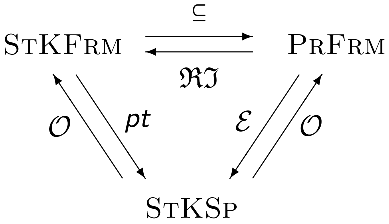
Note Any proximity frame is proximity isomorphic to its frame of round ideals via $\twoheaddownarrow$ and $\bigvee$. But this is not a frame isomorphism. The odd behavior is due to having $*$ for composition.

We have two legs of a triangle, and can figure out half of the third using commutativity.



IV. First equivalences

We complete this using the ends of a proximity frame, which are the meet-prime elements of the lattice of round filters under inclusion.



IV. First equivalences

Drawbacks There are some shortcoming.

1. This does not extend the de Vries dualities!
2. The functors $\subseteq$ and $\mathcal{O}$ are "wrong".
3. Isomorphisms are "funny".
4. It doesn't capture closely the topology.

Still, these are equivalences, and of some interest.

V. Equivalences via regularization

Definition In a proximity frame L, let $ka = \bigwedge \twoheaduparrow a$ and set

$$ja = \bigwedge\{(a \to kb) \to kb : b \in L\}.$$

Call a proximity frame regular if $ja = a$ for each $a \in L$.

Definition $\mathrm{R}\textsc{PrFrm}$ is the category of regular proximity frames.

V. Equivalences via regularization

Proposition For L a proximity frame and L_j its fixed points under j

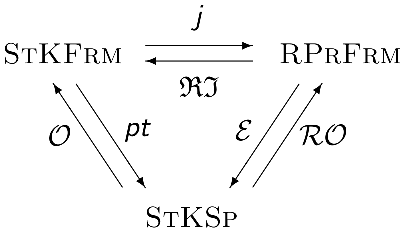
1. L_j is a regular proximity frame.
2. L and L_j are proximity isomorphic via $1, j$.

Theorem There are equivalences

$$\textsc{StKFrm} \underset{\mathfrak{RI}}{\overset{\subseteq}{\rightleftarrows}} \textsc{PrFrm} \underset{\supseteq}{\overset{j}{\rightleftarrows}} \textsc{RPrFrm}$$

V. Equivalences via regularization

We then obtain the following.



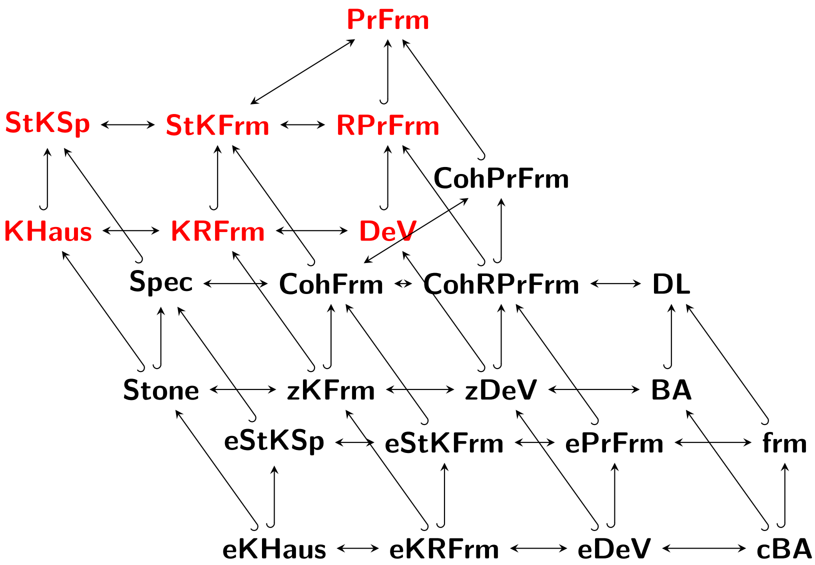
V. Equivalences via regularization

Advantages of these equivalences.

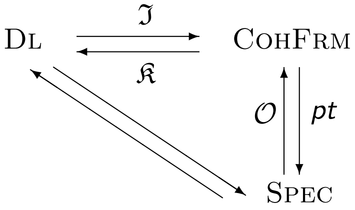
1. They extend the de Vries situation.
2. Isomorphisms in $\mathrm{RP{\small R}F{\small RM}}$ are ordinary bijections.
3. Regularization j extends Booleanization $\mathfrak{B} = \neg\neg$.
4. The composite $\mathcal{RO} = j \circ \mathcal{O}$ captures the topology. In fact $\mathcal{RO}$ gives the sets that are regular in the sense that $A = \mathrm{I}_\tau \mathrm{C}_\pi A$.

VI. Special cases

There are many special cases of these equivalences. Guram has made the following diagram.


PrFrm
StKSp
StKFrm
RPrFrm
CohPrFrm
KHaus
KRFrm
DeV
Spec
CohFrm
CohRPrFrm
DL
Stone
zKFrm
zDeV
BA
eStKSp
eStKFrm
ePrFrm
frm
eKHaus
eKRFrm
eDeV
cBA

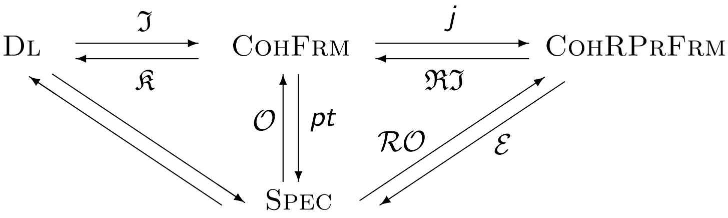
Lets just consider bounded distributive lattices DL. The following equivalences are well known.



Here $\mathfrak{I}$ is the ideal functor, $\mathfrak{K}$ the compact element functor.

VI. Special cases

As spectral spaces are stably compact, our results give



Here coherent regular proximity frames are those regular proximity frames where $a \prec b$ implies there is c with $a \prec c \prec c \prec b$.

VI. Special cases

Here the composite $j \circ \mathfrak{I}$ plays a special role. It gives the frame of distributive ideals in the sense of Bruns-Lakser.

$$\textsc{Dl} \underset{\mathfrak{K}}{\overset{\mathfrak{I}}{\rightleftarrows}} \textsc{CohFrm} \underset{\mathfrak{RI}}{\overset{j}{\rightleftarrows}} \textsc{CohRPrFrm}$$

These distributive ideals were the starting point for our studies. They live in the setting of semilattices, so perhaps there is a more general setting for things still.

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding