Algebras in type-2 fuzzy sets

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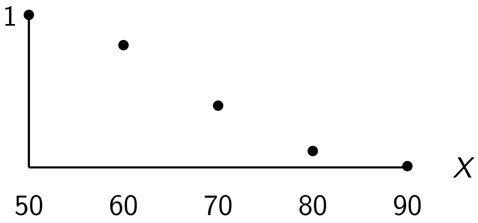
Type-1 fuzzy sets

$X = \{50, 60, 70, 80, 90\}$

A type-1 fuzzy subset of X is a map $\textsc{Cold} : X \to [0,1]$

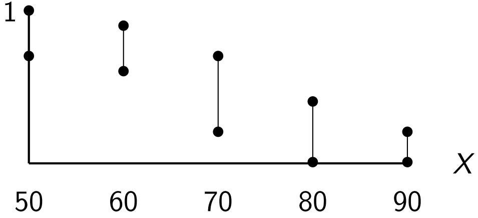
The expert's belief that 60 is cold is 0.8.



Interval valued fuzzy sets

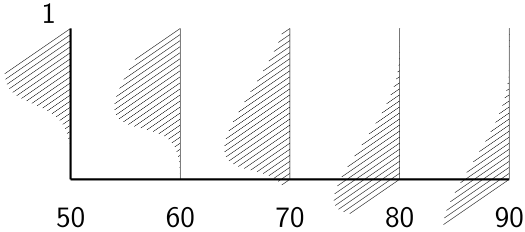
This is a map $\textsc{Cold} : X \to \{(a,b) \in [0,1]^2 : a \leq b\}$.

The expert's belief that 60 is cold is between [0.6,0.9].



Type-2 fuzzy sets

A type-2 fuzzy subset is $\textsc{Cold} : X \to \{f \mid f : [0,1] \to [0,1]\}$



Truth value algebras

The truth value algebras for fuzzy sets, interval valued fuzzy sets, and type-2 fuzzy sets are

$$\mathbf{I} = [0,1]$$

$$\mathbf{I}^{[2]} = \{(a,b) : a \leq b \in \mathbf{I}\}$$

$$\mathrm{M} = \{f \mid f : \mathbf{I} \to \mathbf{I}\}$$

$\mathbf{I}$ and $\mathbf{I}^{[2]}$ sit in M as characteristic functions of points and intervals

Operations

$\mathbb{I}$ and $\mathbb{I}^{[2]}$ are De Morgan algebras. One also considers t-norms and conorms on these.

Definition (Zadeh) Define the following operations on $\mathbb{M}$

1. $(f \sqcap g)(x) = \bigvee \{(f(y) \wedge g(z)) : y \wedge z = x\}$
2. $(f \sqcap g)(x) = \bigvee \{(f(y) \wedge g(z)) : y \vee z = x\}$
3. $f^*(x) = f(1 - x)$
4. $0(x) = 1$ if $x = 0$ and 0 otherwise
5. $1(x) = 1$ if $x = 1$ and 0 otherwise

These are convolutions of the corresponding operations on $\mathbb{I}$. We can also convolute t-norms $\triangle$ and conorms on $\mathbb{I}$.

Equations

Theorem M satisfies the equations for De Morgan algebras except that absorption and distributivity are weakened to the following.

1. $x \sqcap (x \sqcup y) = x \sqcup (x \sqcap y)$
2. $(x \sqcap y) \sqcup (x \sqcap z) \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \sqcap (y \sqcup z)$

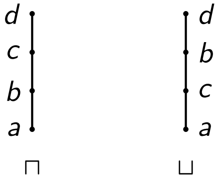
M is not a lattice.

The unbalanced distributive laws do not hold.

M is a type of thing known as a De Morgan Birkhoff system.

Equations

Theorem The variety $V(\mathrm{M})$ is generated by a finite algebra. The variety generated by the reduct $(\mathrm{M}, \sqcap, \sqcup)$ is generated by

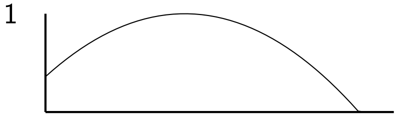


Proof $V(\mathrm{M})$ is generated by the complex algebra of any bounded chain with involution that has at least 5 elements.

So these varieties have solvable free word problems. We do not know if they are finitely based.

A related algebra

Definition A function $f : \mathbf{I} \to \mathbf{I}$ is convex normal if it goes up to 1, then down.



Convex normal functions are a not too restrictive setting for our desired use as belief functions.

A related algebra

Theorem The convex normal functions are a subalgebra of M. For the quotient L of this subalgebra modulo agreement c.a.e.

1. L is a complete, completely distributive DeMorgan algebra
2. L is a compact Hausdorff topological algebra
3. $\int_0^1 |f(x) - g(x)|\, dx$ is a metric on it

Further, the convolution $\triangle$ of any continuous t-norm on I gives a commutative quantale structure $(\mathsf{L}, \triangle, \bigvee)$.

A purpose

Aim: extend the theory of fuzzy controllers to the type-2 setting.

An example

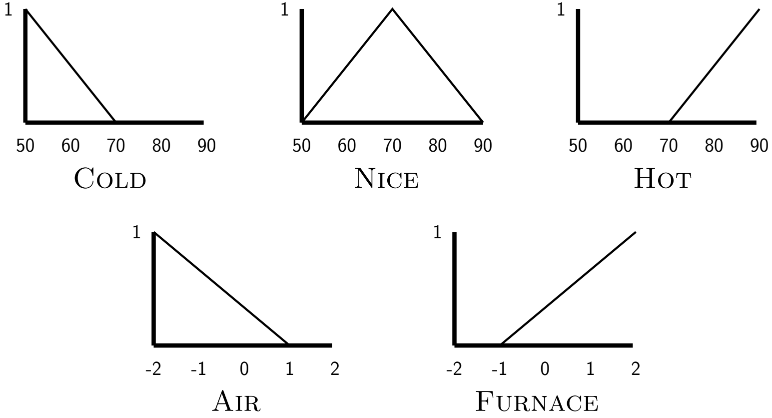
We have a room with a device in it to heat and cool the room and a sensor that measures approximate temperature. Our controller is to adjust the setting of the device.

$$X = \{50, 60, 70, 80, 90\} \quad \text{possible temperatures}$$
$$Y = \{-2, -1, 0, +1, +2\} \quad \text{settings of the device}$$

A setting of -2 puts lots of cold air in the room, $+2$ lots of hot air.

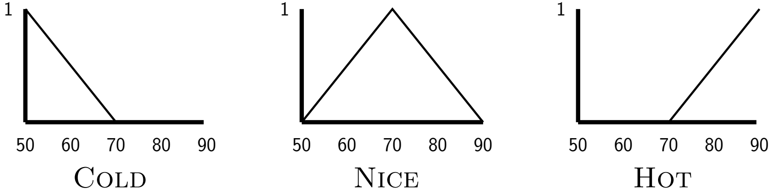
Type-1 fuzzy controllers

Make linguistic variables COLD, NICE, and HOT for temperature; AIR and FURNACE for settings. Experts give fuzzy sets for these.



Type-1 fuzzy controllers

We represent the fuzzy sets for temperature as a matrix

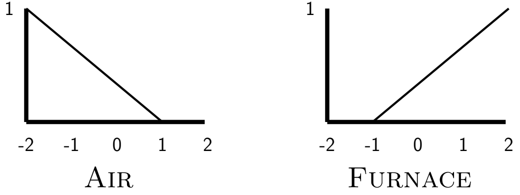


$$P = \begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix}$$

	50	60	70	80	90
Cold	1	.5	0	0	0
Nice	0	.5	1	.5	0
Hot	0	0	0	.5	1

Type-1 fuzzy controllers

And do the same for adjustments



$$Q = \begin{pmatrix} 1 & .7 & .3 & 0 & 0 \\ 0 & 0 & .3 & .7 & 1 \end{pmatrix}$$

	-2	-1	0	1	2
AIR	1	.7	.3	0	0
FURNACE	0	0	.3	.7	1

Type-1 fuzzy controllers

We are given a rule base that says what should be done in each case.

$$R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

	Cold	Nice	Hot
Air	0	1	1
Furnace	1	0	0

Type-1 fuzzy controllers

Then if our sensor gives a reading of 80 for temperature we make a column vector $\hat{T}$ with a 1 in the spot for 80 and 0's elsewhere and compute $Q^T RP(\hat{T})$

$$\begin{pmatrix} 1 & 0 \\ .7 & 0 \\ .3 & .3 \\ 0 & .7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .5 \\ .3 \\ .2 \\ 0 \\ 0 \end{pmatrix}$$

The result is a fuzzy subset of $Y = \{-2, -1, 0, 1, 2\}$ that we then "defuzzify" to get an adjustment to the device.

Type-1 fuzzy controllers

Matrix multiplication computes entries as sums of products.

This multiplication was done using $\cdot$ as product and $\bigvee$ as sum. It can be done using any continuous t-norm $\triangle$ as product and $\bigvee$ as sum. This requires

$$x \triangle \bigvee y_i \;=\; \bigvee (x \triangle y_i)$$

to obtain associativity of matrix multiplication.

Symmetric monoidal categories

Ordinary fuzzy controllers live in the symmetric monoidal category of matrices over $(\mathsf{I}, \triangle, \bigvee)$.

Objects: sets
Morphisms: matrices composed by multiplication

Tesnor product is ordinary product of sets and Kronecker products of matrices. It allows to have more dependent or independent variables in the controller.

Type-2 fuzzy controllers

Do exactly the same with the category of matrices over $(\mathsf{L}, \triangle, \bigvee)$.

Practicality

Implementations would require some restriction on the functions $f : \mathrm{I} \to \mathrm{I}$ (taking n values, or with n linear pieces)

Algorithms for $\sqcap$, $\sqcup$ of convex normal functions are linear in n.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding