Lattice Theory Lecture 1

Basics

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References

Some useful books on lattice theory

Birkhoff: Lattice Theory

Crawley and Dilworth: The Algebraic Theory of Lattices

Balbes and Dwinger: Distributive Lattices

Davey and Priestley: Introduction to Lattices and Order

Birkhoff is older and advanced, and dated, but it is still the best. It conveys what lattice theory is — pervasive. The others are all good as companions and to learn details. Grätzer is encyclopedic.

Background

The prime feature of lattice theory is its versatility. It connects many areas. Algebra, analysis, topology, logic, computer science, combinatorics, linear algebra, geometry, category theory, probability.

We will touch on all these topics. If you know a couple, great, you will learn a little about the others.

Partial orders

Definition A partially ordered set or poset, is a set P with a binary relation $\leq$ on P that satisfies for all $x, y, z \in P$

1. reflexive: $x \leq x$
2. anti-symmetric: $x \leq y$ and $y \leq x \Rightarrow x = y$
3. transitive: $x \leq y$ and $y \leq z \Rightarrow x \leq z$

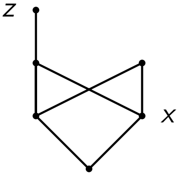
If it additionally satisfies

4. $x \leq y$ or $y \leq z$

the poset is called a liner order or a chain.

Hasse diagrams

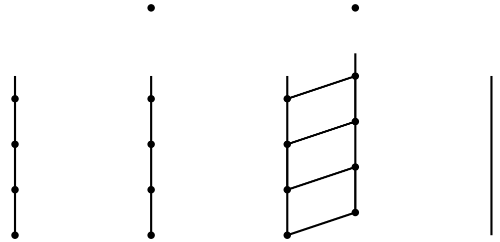
We draw pictures of finite posets using dots for the elements of P and having $x \leq z$ if there are upward line segments from x to z.



We don't include the line from x to z because it is implied by our drawing a picture of a transitive relation.

Hasse diagrams

We use these informally with some infinite posets too. They work not badly for $\mathbb{N}$, but its hard to see the difference between $\mathbb{Q}$ and $\mathbb{R}$ this way.



$\mathbb{N}$ $\qquad$ $\mathbb{N} \cup \{\infty\}$ $\qquad\qquad\qquad$ $\mathbb{Q}$? $\mathbb{R}$?

Terminology

zero 0: The least element of P if there is one.

one 1: The largest element of P if there is one.

bounds: 0 and 1

cover: b covers a if $a < b$ and there is nothing in between.

atom: An element that covers 0.

coatom: An element that is covered by 1

order dual: The poset formed by turning P upside down.

Terminology

For $A \subseteq P$ we say

x is maximal in A: if $x \in A$ and there is no $y \in A$ with $x < y$.

x is a maximum of A: if $x \in A$ and $y \leq x$ for all $y \in A$

upper bounds $U(A)$ of A: all u with $x \leq u$ for all $x \in A$

lower bounds $L(A)$ of A: all v with $v \leq x$ for all $x \in A$

least upper bound $\bigvee A$ of A: a minimum element of $U(A)$

greatest lower bound $\bigwedge A$ of A: a maximum element of $L(A)$

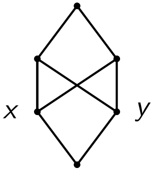
Lattices

Definition A poset L is a lattice if every two element set $\{a, b\}$ has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. It is a complete lattice if every subset $A \subseteq L$ has a least upper bound $\bigvee A$ and a greatest lower bound $\bigwedge A$.

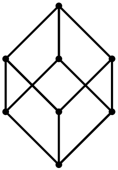
We call $a \vee b$ and $\bigvee A$ the join of a, b or of A
We call $a \wedge b$ and $\bigwedge A$ the meet of a, b or of A

Note if they exist, $\bigvee \varnothing = 0$ and $\bigwedge \varnothing = 1$

Example



not a lattice



lattice

Example The real unit interval $[0,1]$ is a complete lattice

Example The rational unit interval $[0,1] \cap \mathbb{Q}$ is a lattice, but not complete. The set $\{r : r^2 \leq 1/2\}$ has no least upper bound.

Completeness of the real unit interval is the essential reason why all of analysis is done using $\mathbb{R}$ and not $\mathbb{Q}$.

Basic result

Theorem If every subset of a poset L has a meet, then every subset of L has a join, hence L is a complete lattice.

Proof Let $A \subseteq L$ and let $x = \bigwedge U(A)$. For each $a \in A$ and $u \in U(A)$ we have $a \leq u$. Thus a is a lower bound of $U(A)$, and since x is the greatest lower bound of $U(A)$ we have $a \leq x$ for each $a \in A$. Thus x is an upper bound of A. So $x \in U(A)$, and since x is a lower bound of $U(A)$ it must be the least element of $U(A)$. Thus $x = \bigvee A$.

Closure systems

Observation The union of a collection of sets is the smallest set that contains them all, and the intersection of a collection of sets is the largest set contained in them all.

Definition A closure system is a collection $\mathcal{C}$ of subsets of a set X where $\mathcal{A} \subseteq \mathcal{C} \Rightarrow \bigcap \mathcal{A} \in \mathcal{C}$.

Corollary If $\mathcal{C}$ is a closure system, then as a poset, every subset $\mathcal{A} \subseteq \mathcal{C}$ has a meet, hence $\mathcal{C}$ is a complete lattice.

Note A dual result holds if $\mathcal{C}$ is a collection of subsets of X closed under unions. We call such a dual closure system.

Examples of lattices

Example 1 The power set $\mathcal{P}(X)$ of a set X is a closure system, hence a complete lattice.

Example 2 The collection $\mathcal{S}(V)$ of subspaces of a vector space V is a closure system, hence a complete lattice.

Example 3 The collection $\mathcal{C}(X)$ of closed sets of a topological space (X, τ) is a closure system.

Example 4 The collection $\mathcal{O}(X)$ of open sets of a topological space (X, τ) is a dual closure system.

Example 5 The collection $\mathcal{I}(R)$ of ideals of a ring R is a closure system, hence a complete lattice.

Examples of lattices

Example 6 The collection $\mathcal{N}(G)$ of normal subgroups of a group G is a closure system, hence a complete lattice.

Exercise What are joins and meets in each of these examples?

Exercise Is the collection of all finite subsets of $\mathbb{N}$ a closure system? Is it a complete lattice? What are joins?

Note, many of these examples lead to extensive areas of study.

Quasi-orders

To describe another key example, we discuss quasi-orders.

Definition A quasi-order on a set X is a relation $\sqsubseteq$ that is reflexive and transitive, but not necessarily anti-symmetric.

Proposition Let $\sqsubseteq$ be a quasi-order on X,

1. setting $x \approx y$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$ gives an equivalence relation
2. setting $x/\approx \leq y/\approx$ iff $x \sqsubseteq y$ gives a partial order on $X/\approx$

Quasi-orders

Example Define $\sqsubseteq$ on the power set $\mathcal{P}(\mathbb{N})$ by

$$S \sqsubseteq T \text{ iff } S, T \text{ differ by a finite set}$$

It turns out that $\mathcal{P}(\mathbb{N})/\approx$ is a lattice. Questions about this lattice lead to some very involved topics in set theory.

Exercise does this lattice have bounds?

Exercise does this lattice have any atoms? any covers?

Quasi-orders

Example Let $\mathcal{L}$ be a first order language, say with one binary relation symbol, and $\mathcal{F}(\mathcal{L})$ be its first order formulae such as

$$\phi : \quad \forall x \exists y \, (xRy \text{ and } x \neq y)$$

Define $\sqsubseteq$ on $\mathcal{F}(\mathcal{L})$ by setting $\phi \sqsubseteq \psi$ if $\phi \to \psi$ is provable.

Theorem The relation $\sqsubseteq$ is a quasi-order with the associated equivalence relation $\approx$ being logical equivalence. With the associated partial order $\mathcal{F}(\mathcal{L})$ is a lattice where

$$\begin{aligned} \phi/\!\approx \;\wedge\; \psi/\approx &= (\phi \text{ and } \psi)/\!\approx \\ \phi/\!\approx \;\vee\; \psi/\approx &= (\phi \text{ or } \psi)/\!\approx \end{aligned}$$

Exercise what are the bottom and top of $\mathcal{F}(\mathcal{L})$? Is it complete?

Connections to category theory

Definition A category $\mathcal{C}$ consists of a collection of objects, a collection of morphisms between objects, and a rule to combine morphisms $f \circ g$ such that

1. each object A has an identity morphism 1_A
2. composition is associative

Proposition A poset P gives a category $\mathcal{C}_P$ whose objects are the elements of P where there is a unique morphism $f : x \to y$ iff $x \leq y$.

Many ideas in category theory have precursors in order theory, and these help to understand the situation.

Connections to category theory

Definition A product of objects A, B in $\mathcal{C}$ is an object $A \times B$ with morphisms to A and B such that any other object with morphisms to A and B factors through $A \times B$. Coproducts are defined dually.

$$\begin{tikzcd}
& C \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& A \times B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

Proposition For a poset P, the category $\mathcal{C}_P$ has finite products and coproducs iff P is a lattice. It has all products and coproducts iff P is a complete lattice.

Interlude

We have defined lattices and indicated that they arise in many areas of math, such as analysis, algebra, linear algebra, set theory, logic, and category theory.

We could spend all week, and more, just giving examples of important ways lattices occur in different areas of math.

The real study of lattice theory is in understanding the range of examples and relating them to another.

To begin this process, we must make a general study of lattices and their basic properties. We now turn to this.

Lattices as algebras

Definition A type τ consists of a set, called an indexing set, and a map $\tau : I \to \mathbb{N}$. We write $\tau(i)$ as n_i when τ is understood.

Definition An algebra of type τ is a pair $(A, (f_i)_I)$ where

1. A is a set
2. $f_i : A^{n_i} \to A$

If $n_i = 0$, then since A^0 is a 1-element set, f_i is determined by its image, so f_i picks one element of A. We call such f_i nullary operations or constants.

Lattices as algebra

Example A group $(G, \cdot, ^{-1}, e)$ is an algebra of type $2, 1, 0$.

Example A ring $(R, +, \cdot, -, 0, 1)$ is an algebra of type $2, 2, 1, 0, 0$.

Aim: lattices correspond to certain algebras $(L, \wedge, \vee)$ of type $2, 2$, and bounded lattices correspond to certain algebras $(L, \wedge, \vee, 0, 1)$ of type $2, 2, 0, 0$.

Lattices as algebras

Proposition For any lattice L, the algebra $(L, \wedge, \vee)$ satisfies

1. $x \wedge x = x$
2. $x \vee x = x$
3. $x \wedge y = y \wedge x$
4. $x \vee y = y \vee x$
5. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
6. $(x \vee y) \vee z = x \vee (y \vee z)$
7. $x \wedge (x \vee y) = x = x \vee (x \wedge y)$

Equation (7) is the only one to relate $\wedge$ to $\vee$. Its called absorption.

Exercise Prove that (5) holds in any lattice?

Lattices as algebras

Proposition For any algebra $(L, \wedge, \vee)$ that satisfies (1)-(7), there is a partial ordering $\leq$ on L given by

$$x \leq y \quad \text{iff} \quad x \wedge y = x$$

With this partial ordering L is a lattice where $x \wedge y$ is the greatest lower bound of x, y and $x \vee y$ is the least upper bound of x, y.

Exercise If $(L, \wedge, \vee)$ satisfies (1)-(7) show $x \wedge y = x$ iff $x \vee y = y$.

Exercise What equations do you need to add to (1)-(7) to capture bounded lattices $(L, \wedge, \vee, 0, 1)$?

Lattices as algebras

We use "lattice" to refer either to a poset where any two element have a glb and lub, or to an algebra satisfying (1)-(7) and pass freely between the two notions.

Often the algebraic view is useful when considering finite joins and meets, while the order-theoretic one is vital when considering any infinite joins and meets that might exist.

Homomorphisms, subalgebras, and products

Homomorphisms between algebras of a given type are maps that are compatible with their operations. So ...

Definition A lattice homomorphism is a map $f : L \to M$ where

1. $f(x \wedge y) = f(x) \wedge f(y)$
2. $f(x \vee y) = f(x) \vee f(y)$.

For bounded lattices, so type 2,2,0,0, also $f(0) = 0$ and $f(1) = 1$

Definition A lattice embedding is a 1-1 lattice homomorphism.

Exercise For a lattice embedding prove $x \leq y$ iff $f(x) \leq f(y)$.

Homomorphisms, subalgebras, and products

A subalgebra of an algebra is a subset that is closed under the operations. So ...

Definition A sublattice of a lattice L is a subset $S \subseteq L$ where

1. $x, y \in S \Rightarrow x \wedge y \in S$
2. $x, y \in S \Rightarrow x \vee y \in S$

For bounded lattices we also need $0 \in S$ and $1 \in S$.

Exercise Is $\{A : 2 \in A\}$ a sublattice of $\mathcal{P}(\mathbb{N})$?

Exercise Is $\{A : 2 \in A\}$ a bounded sublattice of $\mathcal{P}(\mathbb{N})$?

Homomorphisms, subalgebras, and products

Definition For lattices L, M their product is the set

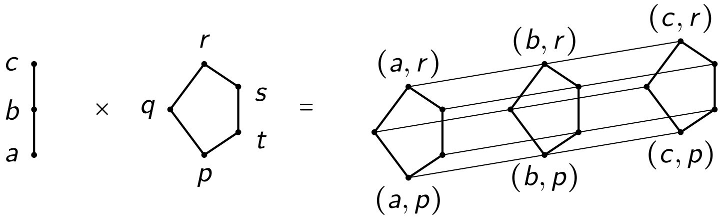
$$L \times M = \{(x, y) : x \in L \text{ and } y \in M\}$$

with joins and meets defined componentwise, so

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$
$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$

Homomorphisms, subalgebras, and products

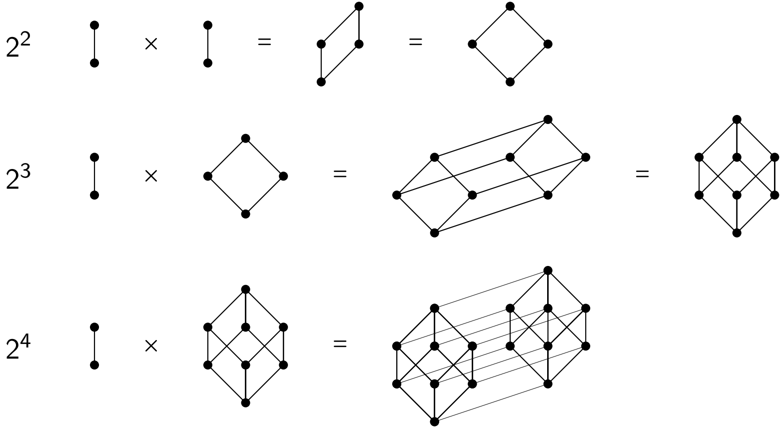
We can draw Hasse diagrams of products in a nice way.



You replace each element of the first lattice with a small copy of the second and order accordingly.

Homomorphisms, subalgebras, and products

Let 2 be the two-element lattice. We consider Hasse diagrams of its finite powers 2^n



Homomorphisms, subalgebras, and products

There is a version for infinite products as well ...

Definition For lattices L_j where $j \in J$, their product is the set

$$\prod_J L_j = \{\alpha | \alpha : J \to \bigcup_J L_j \text{ where } \alpha(j) \in L_j \text{ for each } j \in J\}$$

with operations componentwise, so $(\alpha \wedge \beta)(j) = (\alpha(j)) \wedge (\beta(j))$.

Exercise If each L_J is complete, prove $\prod L_j$ is complete.

Quotients

Definition A congruence $\approx$ on a lattice L is an equivalence relation where

1. $x \approx u$ and $y \approx v \Rightarrow x \wedge y \approx u \wedge v$
2. $x \approx u$ and $y \approx v \Rightarrow x \vee y \approx u \vee v$

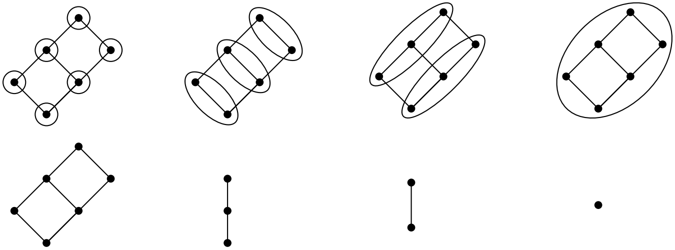
Proposition If $\approx$ is a congruence on L then $L/\approx$ is a lattice where

$$(x/\approx) \wedge (y/\approx) = (x \wedge y)/\approx$$
$$(x/\approx) \vee (y/\approx) = (x \vee y)/\approx$$

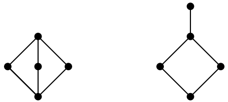
Further, $f : L \to L/\approx$ given by $f(x) = x/\approx$ is an onto lattice homomorphism, and up to isomorphism, each such arises this way.

Quotients

Example Congruences on L and the resulting quotients are ...



Exercise Repeat this for the lattices



Varieties

Definition A class V of algebras is a variety if it is all algebras of a given type that satisfy some set of equations.

Examples Groups, abelian groups, rings, commutative rings, real vector spaces, and lattices are varieties.

Theorem A class of algebras is a variety iff it is closed under HSP, homomorphic images, subalgebras, and products.

Varieties

There are infinitely many subvarieties of lattices, but two of particular importance.

Definition For a lattice L these are equivalent.

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, z

The lattices that satisfy these are called distributive. Distributive lattices form a variety.

Note These are familiar when $\wedge, \vee$ are interpreted as “and, or” of logic, or as intersection and union of sets.

Varieties

Definition For a lattice L these are equivalent. For all x, y, z

1. $z \leq x \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
2. $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
3. $(x \vee z) \wedge (y \vee z) = ((x \vee z) \wedge y) \vee z$
4. $(x \wedge z) \vee (y \wedge z) = ((x \wedge z) \vee y) \wedge z$

The lattices that satisfy these are called modular. Modular lattices form a variety.

Note Modularity says that distributivity holds for x, y, z when there is a comparability among them.

Varieties

The varieties of all lattices, of distributive lattices, and modular lattices induce a split in lattice theory like that of commutative and non-commutative in algebra. Roughly ...

distributive	logic and set theory
modular	algebra, combinatorics, geometry

More than anyone else, these were due to Dedekind $\approx$ 1880's, but distributive lattices in some form existed since Boole in the 1850s.

Lattices with additional operations

Many classes of interest will be varieties of lattices with additional operations.

Definition An algebra $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra if

1. $(B, \wedge, \vee, 0, 1)$ is a bounded lattice
2. $(B, \wedge, \vee)$ is distributive
3. $x \wedge x' = 0$
4. $x \vee x' = 1$

Boolean algebras form a variety.

Lattices with additional operations

Definition An algebra $(H, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra if

1. $(H, \wedge, \vee, 0, 1)$ is a bounded lattice
2. $(H, \wedge, \vee)$ is distributive
3. $x \to x = 1$
4. $x \wedge (x \to y) = x \wedge y$
5. $y \wedge (x \to y) = y$
6. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

Heyting algebras form a variety.

Items (3)-(6) are equivalent to $x \to y$ being the largest element whose meet with x is beneath y. This description is not obviously equational though.

Lattices with additional operations

Definition An algebra $(B, \Diamond, \wedge, \vee, ', 0, 1)$ is a modal algebra if

1. $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra
2. $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$
3. $\Diamond 0 = 0$

Modal algebras form a variety.

If we view a Boolean algebra as modeling logical propositions, so $x \wedge y$ is seen as x AND y, then $\Diamond x$ may be POSSIBLY x.

There are many interesting subvarities of modal algebras obtained by adding equations to the list such as $\Diamond\Diamond x = \Diamond x$.

Definition $(B, \circ, \smile, \triangle, \wedge, \vee, ', 0, 1)$ is a relation algebra if

1. $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra
2. $x \circ (y \circ z) = (x \circ y) \circ z$
3. $x \circ \triangle = x = \triangle \circ x$
4. $(x \vee y) \circ x = (x \circ z) \vee (y \circ z)$
5. $(x \circ y)^{\smile} = y^{\smile} \circ x^{\smile}$
6. $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$
7. $(x^{\smile} \circ (x \circ y)') \vee y' = y'$

Relation algebras form a variety. They model the following:

A binary relation R on a set X is a subset $R \subseteq X \times X$. The set of all R is the power set $B = \mathcal{P}(X \times X)$. Compose relations via $\circ$, have the converse relation $R^{\smile}$ and identity relation $\triangle$.

Concluding remarks

Lattices arise in many areas of mathematics.

Many classes of lattices that are of interest form varieties of lattices, or varieties of lattices with additional operations.

There are also many classes of lattices of interest that do not form varieties, such as chains, algebraic, and continuous lattices.

Next time we resume with a study of distributive lattices.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding