Orthodomular Structures and Categories

OR

Everything Old is New Again

John Harding

New Mexico State University
www.math.nmsu.edu/∼JohnHarding.html

jharding@nmsu.edu

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Overview

We discuss several related topics.

Much of this is recent, but has ties to very old topics.

These are results of many people, some mine, some others.

Part I

Categorical quantum mechanics

The basic idea

In quantum logic one uses a structure (OMP) to model a single quantum system. Abramsky and Coecke use a structure (category) to model all quantum systems simultaneously.

- objects of the category = quantum systems
- morphisms of the category = processes

Of course, certain features are required of the category.

Strongly closed categories

Assume $\mathcal{C}$ is a category with

- a symmetric monoidal tensor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- an involution $\dagger : \mathcal{C} \to \mathcal{C}$ that is the identity on objects
- biproducts $A \oplus B$ (means it is a product and coproduct)

Modulo small details, this gives a DSMB — a dagger symmetric monoidal category with biproducts.

Adding dual objects A^* and morphisms $\epsilon_A : A \otimes A^* \to I$ to the tensor unit I gives a strongly compact closed category.

Examples

Examples of strongly compact closed categories with biproducts.

Rel sets as objects, relations as morphisms.

$\mathbf{Mat_K}$ natural numbers as objects, matrices over K as morphisms.

FDHilb finite dim Hilbert spaces with linear maps as morphisms.

$\mathbf{VBund}(\mathbf{X})$ finite dim vector bundles over X.

Remark All these examples were considered explicitly, or implicitly, by Foulis and his grandson Crown.

Features

For such a category $\mathcal{C}$ Abramsky and Coecke

- build a semiring of scalars as the endo's of the tensor unit
- develop inner products, measurements, probabilities, unitaries
- make ties to linear logic
- develop a graphical calculus for analyzing complex protocols

This is all very interesting, and doesn't require a lot of category theory background to learn.

Quantum logic in this setting

The categorical approach by nature deals with interactions between systems. Single systems are just represented by abstract objects in a category.

Still, we can associate an orthomodular structure to each object in such a category as a way to study fine details of single systems.

This enriches things and allows for finer analysis of the axioms needed. We can import 50+ years of quantum logic.

Attaching orthomodular structures to objects

Suppose $\mathcal{C}$ is a DSMB and A is an object in $\mathcal{C}$

Build a *Dec A* from the biproduct decomp's $A \to A_1 \oplus A_2$.

(We give details in a few minutes)

Theorem *Dec A* is an OA. Its an OMP if idempotents strongly split

Quantum logic in the categorical setting

Parts of quantum logic come for free ...

- An orthomodular structure *Dec A* for each system A
- A quasi-ordered semiring of scalars S
- States $\sigma : Dec\ A \rightarrow [0,1]$ into the unit interval of S.
- Probabilities of *Yes* outcome to measurement in given state

These work somewhat as you would hope, but not completely. Lets examine the tensor products the categorical approach is built for.

Tensor products

Theorem For A, B objects in $\mathcal{C}$ and σ, ϕ "normal states"

1. There is a bimorphism $\otimes : Dec\ A \times Dec\ B \to Dec\ (A \otimes B)$
2. Exists a normal χ on $Dec\ (A \otimes B)$ with $\chi(p \otimes q) = \sigma(p)\phi(q)$

Remark Other key properties of tensor products do not follow.

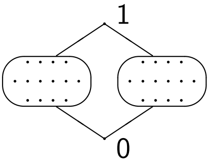
- $Dec\ (A \otimes B)$ is not generated by the images of $Dec\ A, Dec\ B$
- No universal mapping property
- Lifting of states applies only for "normal" ones.

An example

In $\mathbf{Mat_K}$ where $K = \mathbb{Z}_2$, objects are $1, 2, 3, \ldots$

Dec $2 =$ 4-element Boolean algebra.

Dec $4 =$ horizontal sum of two 16-element Boolean algebras.



The tensor product $\otimes$: *Dec* $2 \times$ *Dec* $2 \rightarrow$ *Dec* 4 is the classical tensor product $2^2 \times 2^2 \rightarrow 2^4$, but equipped with a phantom 16-element Boolean algebra not connected to either subsystem.

Summary for Part I

This categorical approach looks promising.

But the axioms need refinement, and incorporating quantum logic is a sharp tool to accomplish this.

This seems very close in spirit to the quantum logic of Mackey, etc.

Part II

Building orthomodular structures from objects in a category

First approach

Basic idea — the idempotents $E(R)$ of a ring R form an OMP.

$$e \leq f \quad iff \quad ef = e = fe \qquad\qquad e' = 1 - e$$

We can easily do with a semiring R, showing how to build an OMP from the idempotent endomorphisms in any semi-additive category, so in any DSMB category as above.

We can get by with far less ...

Foulis semigroups

For $e \in E(R)$ the annihilator $\{x : ex = 0\} = (1 - e)R$.

Having an involution $*$ gives a bit of commutativity for self-adjoint idempotents, $ef = e$ iff $fe = e$, and this lets us recover $1 - e$ from the annihilator ideal.

Defn A Foulis semigroup is a semigroup S with involution $*$ where each $a \in S$ has a self-adjoint idempotent a' with $\{x : ax = 0\} = a'S$.

Theorem $\{a' : a \in S\}$ forms an OML under the above order with $'$ as orthocomplement, and each OML arises this way.

Kernels

In a category with 0, a kernel of $a : A \to B$ is $k : K \to A$ where

$$\{x : ax = 0\} = k\mathcal{C}(\cdot, K)$$

Theorem In a category $\mathcal{C}$ with dagger $\dagger$ and 0 and enough well behaved kernels, the endomorphsims of any object form a Foulis semigroup, so give an OML.

Example A Foulis semigroup as a 1-element category.

Example The category of OMLs with Galois connections as maps.

This was first done by Crown (1980), more recently by Heunen and Jacobs with many interesting ties to categorical logic.

Partial Semigroups

We can do with far less still ...

Defn An OPS is a commutative, idempotent, partial semigroup S with 0 where each $a \in S$ has $a' \in S$ with

1. $aa' = 0$
2. ab defined $\Rightarrow a'b$ defined
3. $ab = ac$ and $a'b = a'c \Rightarrow b = c$

Theorem OPSs are the commuting meet fragments of OMPs.

This was done by Gudder and Schelp. It gives a very general way to build OMPs from a set S of idempotent endomorphisms of $A \in \mathcal{C}$.

Second approach

Theorem Suppose $\mathcal{C}$ is a category with finite products where projections are epic and each diagram below is a pushout.

$$
\begin{array}{ccccc}
 & & P \times Q \times R & & \\
 & \swarrow & & \searrow & \\
P \times Q & & & & Q \times R \\
 & \searrow & & \swarrow & \\
 & & Q & &
\end{array}
$$

For $A \in \mathcal{C}$ the binary product decomp's $A \simeq P \times Q$ form an OA.

$$
\begin{aligned}
A \simeq P \times (Q \times R) \quad &\leq \quad A \simeq (P \times Q) \times R \\
(A \simeq P \times Q)^{\perp} \quad &= \quad A \simeq Q \times P
\end{aligned}
$$

Summary for Part II

Either approach (via endomorphisms of A or decomp's of A) can be used in the setting of a DSMB category in Part I.

It seems interesting to see how categorical conditions affect the orthostructures produced. This is maybe useful in considering versions of categorical quantum mechanics.

I'm biased, but rather like the decomp's.

Part III

The topos approach of Isham and Butterfield

Topos approach

Very briefly ...

This approach takes the projection lattice $L = \mathcal{L}(\mathcal{H})$ and sets

$$\mathrm{BSub}(L) = \{B : B \text{ is a Boolean subalgebra of } L\}$$

Then $\mathrm{BSub}(L)$ is a poset under set inclusion. Works for any OML.

The topos approach

As with any poset, $\mathrm{BSub}(L)$ is a category.

Isham and Butterfield consider presheaves $V : \mathrm{BSub}(L) \to \mathbf{Set}$ to be used as generalized valuations, contexts, and so forth.

One result shows a certain presheaf failing to have a global element is equivalent to the Kochen Specker theorem.

$\mathrm{BSub}(L)$, and similar structures, feature in approaches to construct a non-commutative Gelfand duality.

Small observations

Recently, Mirko Navara and I made some small observations ...

Theorem Each OML L is determined by $\mathrm{BSub}(L)$.

More is true. Isomorphisms between $\mathrm{BSub}(L)$ and $\mathrm{BSub}(M)$ are uniquely determined by isomorphisms between L and M provided all blocks have more than 4 elements.

Small observations

This yields a categorical result that likely is just a curiosity.

Theorem The category of OMLs having no 4-element blocks with morphisms being onto homomorphisms is dually equivalent to a subcategory of algebraic lattices and order preserving maps.

Proof Define a contravariant $F : \mathbf{OML} \to \mathbf{Alg}$ by sending L to $\mathrm{BSub}(L)$ and sending a homomorphism f to f^{-1}.

Summary for Part III

Knowing $\text{BSub}(L)$ encodes L gives some perspective on this topos approach. This is also worth considering if we use $\text{BSub}(L)$ as part of a duality theory.

The internal logic in this topos is a Heyting algebra built from the downsets of the poset $\text{BSub}(L)$. This is a highly non-trivial object.

Question Is a C^* algebra determined by $\text{AbSub}(C)$?

Part IV

Bits and pieces

Some categorical things from years past ...

Kalmbach's construction $\mathcal{K}$ is left adjoint to the forgetful functor from OMPs to Posets.

Bruns and Roddy characterized the projective Boolean algebras in OML as the countable ones. They also showed MO_2 is s-projective.

OMLs have amalgamation over Boolean subalgebras.

Remark There is likely quite a bit of unfinished work in these lines that may be worthwhile to pursue. One specific question ...

Question Are epi's surjective for OMLs?

References

These are convenient sources to gain further info on things discussed here.

- My homepage www.math.nmsu.edu/∼jharding
- Bob Coecke's homepage (Google it)
- Chris Heunen's homepage (Google it)
- Archiv for papers by Isham and Butterfield, also search Google for Topos and quantum mechanics

Thanks to the organizers

Thank you for listening.

Papers at www.math.nmsu.edu/$\sim$jharding