Automorphisms of Decompositions

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Overview

This talk describes a piece in my long term program that lies between the Hilbert space approach via the OML $\mathcal{C}(\mathcal{H})$ and the quantum logic approach via general OMPs etc.

This program replaces $\mathcal{C}(\mathcal{H})$ with a structure Fact X built from the direct product decompositions of an object X.

I Background on Fact X

II Wigner's theorem

III Wigner's theorem for sets

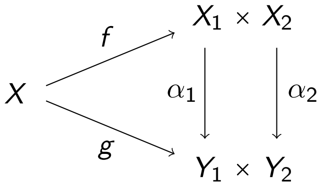
IV Further remarks

I Background on Fact X

A binary direct product decomposition of a set X consists of sets X_1 and X_2 and an isomorphism (bijection)

$$f : X \to X_1 \times X_2$$

Two such binary decompositions are equivalent if there are isomorphisms α_1, α_2 making the following diagram commute.



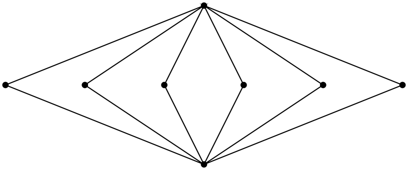
I Background on Fact X

Definition Let Fact X be the collection of all equivalence classes of binary direct product decompositions of X. On this, define

1. $[X \simeq X_1 \times X_2]^\perp = [X \simeq X_2 \times X_1]$
2. $[X \simeq X_1 \times (X_2 \times X_3)] \leq [X \simeq (X_1 \times X_2) \times X_3]$

Theorem For a set X, Fact X is an OMP.

Example If $|X| = 4$, Fact X is as follows:



If $|X|$ is the product of n primes, its blocks have n atoms.

If $|X| = 27$, Fact X has 5,001,134,190,558,105,600,000 atoms.

I Background on Fact X

This construction applies in many other settings, such as groups, rings, vector spaces, topological spaces, etc.

For a Hilbert space, Fact $\mathcal{H} \simeq \mathcal{C}(\mathcal{H})$.

It applies to objects in "good" categories like biproduct categories, or ones with finite products where these are pushouts

$$\begin{array}{ccc} X_1 \times X_2 \times X_3 & \longrightarrow & X_2 \times X_3 \\ \downarrow & & \downarrow \\ X_1 \times X_2 & \longrightarrow & X_2 \end{array}$$

It allows settings close to Hilbert space like Hermitian vector bundles or normed groups with operators, to more exotic ones.

I Background on Fact X

I have been developing aspects of quantum mechanics using Fact X in place of $\mathcal{C}(\mathcal{H})$.

- propositions
- observables
- states in specific cases
- probabilities
- categorical versions

Today a version of Wigner's theorem

II Wigner's Theorem

Representations of a group G as symmetries of a system modeled by a structure X are physically motivated (alá Wigner) as group homomorphisms

$$\pi : G \to \mathrm{Aut}(\mathrm{Fact}\, X)$$

A Wigner's theorem aims to describe the automorphisms of $\mathrm{Fact}\, X$ in terms of automorphisms of X.

Aim A version of Wigner's theorem for sets.

II Wigner's Theorem

A first result is easy and general. For any structure X, there is a group homomorphism

$$\Gamma : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Fact}\, X)$$

This map is usually neither one-one or onto. For a Hilbert space $\mathcal{H}$ with dim $\mathcal{H} \geq 3$, Wigner's theorem shows that

$$\ker \Gamma = \{z\,\mathbf{I} : z \in \mathbb{C} \text{ and } |z| = 1\}$$
$$\mathrm{Im}\ \Gamma = \text{ a subgroup of index 2}$$

One needs also anti-unitaries of $\mathcal{H}$ to get onto. Both defects cause complications with group representations.

III Wigner's Theorem for sets

"Conjecture" For a set X, the map $\Gamma : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Fact}\, X)$ is a group isomorphism.

Previously known

- if $|X| = pq$ is the product of two primes, this is false.
- if $|X| = 8$ this is false.
- if $|X| = 27$ this is true! (with Tim Hannan)

The first item is expected, like the exception when $\dim \mathcal{H} = 2$. The second item was not easy and was not encouraging. The third item is a 30 page proof, but obviously limited in scope.

III Wigner's Theorem for sets

Theorem For X an infinite set, $\Gamma : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Fact}\, X)$ is a group isomorphism.

Sketch of the proof:

It is quite difficult, some 50 pages. Its structure is like many proofs about $\mathrm{Fact}\, \mathcal{H}$ for a Hilbert space — push things down to height 3, solve it, lift it up.

Let the size of a decomposition $[X \simeq X_1 \times X_2]$ be $|X_1|$.

Step 1 automorphisms of $\mathrm{Fact}\, X$ preserve size of decompositions

Step 2 atoms of $\mathrm{Fact}\, X$ are decompositions whose size is prime

Step 3 a decomposition of infinite size is a join of ones of size 3

III Wigner's Theorem for sets

This says that automorphisms of Fact X are determined by their action on decompositions of size 3.

Step 4 the interval beneath a decomposition $[X \simeq X_1 \times X_2]$ is isomorphic to Fact X_1 (with Taewon Yang)

Step 5 any two decompositions of size 3 can be connected by a finite sequence of intervals beneath decompositions of size 27.

These results let us use the result that automorphisms of Fact Y for a 27-element set Y are given by permutations of the set. $\square$

IV Further remarks

My current aim is to add group representations to the program of treating quantum systems via Fact X. Some progress ...

Definition For a category $\mathcal{C}$ and group G, the functor category $\mathcal{C}^G$ consists of objects X of $\mathcal{C}$ with a representation $\pi : G \to \mathrm{Aut}(X)$.

Theorem If $\mathcal{C}$ is good, so is $\mathcal{C}^G$. So Fact (X, π) is an OA

Easy modifications give similar results for dagger categories and unitary representations.

IV Further remarks

A first place to start is $\mathcal{C}^{\mathbb{R}}$.

- Objects (X, π) consist of a structure X with $\pi : \mathbb{R} \to \mathrm{Aut}(X)$.
- Call π the natural frequency of X.
- For Hilbert spaces, one usually chooses $\pi_t(v) = e^{it} v$

With previous results about observables in the setting of Fact X, one gets a sort of time independent version of a Schrödinger equation from an observable H and natural frequency π

IV Further remarks

If H takes finitely many values $\lambda_1, \ldots, \lambda_n$, then H gives an n-ary decomposition $X \simeq X_1 \times \cdots \times X_5$

The generalized Schrödinger's equation $U_t = \pi_{Ht}$ (see $U_t = e^{iHt}$) gives

$$U_t(x_1, \ldots, x_n) = (\, \pi_{\lambda_1 t}(x_1), \ldots, \pi_{\lambda_n t}(x_n)\,)$$

When H has infinitely many outcomes I think there is a similar version involving sheaf representations over Stone spaces, but haven't settled all the details yet.

IV Further remarks

Finally, what can we say of the natural frequencies π of sets?

The subdirectly irreducible ones are the Prüfer p^∞ groups, analogs of $\{z \in \mathbb{C} : |z| = 1\}$ where elements have orders p^k.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding