Completions of Ordered Algebraic Structures — A Survey

John Harding

New Mexico State University
www.math.nmsu.edu/JohnHarding.html

jharding@nmsu.edu

UncLog-2008
JAIST, March 2008

This is a survey of results by many people including ...

Bezhanishvili, Dilworth, Dunn, Gehrke, Givant, Harding, Hartung, Jónsson, Monk, Nagahashi, Sahlqvist, and Venema.

Introduction

The ordered field of rationals $(\mathbb{Q}, \leq, +, \cdot)$ is an ordered algebraic structure — i.e. a poset with additional operations.

In 1880, Dedekind constructed the reals $(\mathbb{R}, \leq, +, \cdot)$ to complete the rationals — that is, to embed the rationals into a structure where every (bounded) subset has a join and meet.

This is a very good completion as it preserves all existing joins and meets as well as many algebraic properties.

Introduction

Ordered algebraic structures play a role in a wide range of areas, including analysis, logic, theoretical computer science, and foundations of physics.

In these applications certain infinite joins and meets often play an important role linked to some kind of infinitary or limit process.

It is common to seek a completion of such an ordered algebraic structure preserving certain joins and meets and certain aspects of the algebraic structure.

Introduction

There are many kinds of ordered structures; Heyting algebras, modal algebras, lattices, Boolean algebras with operators (BAO's), *l*-groups, residuated lattices, orthomodular lattices (OML's), ...

There are many kinds of completions; MacNeille completions, ideal completions, filter completions, canonical completions, ...

There are many isolated results ...

Aim First steps toward a unified theory of completions.

Basic Definitions

Defn For a poset P, we say $P \leq C$ is a completion if

- $P \leq C$ is an order embedding
- C is a complete lattice

Defn We say $P \leq C$ is

- join-dense if $c \in C \Rightarrow c = \bigvee\{p \in P : p \leq c\}$
- meet-dense if $c \in C \Rightarrow c = \bigwedge\{p \in P : c \leq p\}$

Lemma

- join-dense $\Rightarrow$ preserves existing meets (meet regular)
- meet-dense $\Rightarrow$ preserves existing joins (join regular)

Standard Completions

Among the commonly encountered completions are

- The downset completion $P \leq \textit{Down } P$
- The ideal completion $P \leq \textit{Ideal } P$
- The filter completion $P \leq \textit{Filter } P$
- The $\sigma-$ideal completion $P \leq \sigma - \textit{Ideal } P$
- The MacNeille completion $P \leq \textit{Norm } P$
- The canonical completion $P \leq P^\sigma$

The canonical completion is a key tool in modal logic. For B a Boolean algebra, B^σ is the power set of the Stone space of B.

Note There are many other completions specific to particular kinds of structures.

Template for Completions

Fact A relation $R \subseteq X \times Y$ gives a Galois connection between the power sets of X and Y. This is called a polarity.

Template for Completions

Fact A relation $R \subseteq X \times Y$ gives a Galois connection between the power sets of X and Y. This is called a polarity.

Suppose

$$\mathcal{I} = \text{a set of downsets of } P \text{ that separates points}$$

$$\mathcal{F} = \text{a set of upsets of } P \text{ that separates points}$$

$$R \subseteq \mathcal{I} \times \mathcal{F} \text{ is given by } I\,R\,F \Leftrightarrow I \cap F \neq \emptyset$$

Template for Completions

Fact A relation $R \subseteq X \times Y$ gives a Galois connection between the power sets of X and Y. This is called a polarity.

Suppose

$$\mathcal{I} = \text{a set of downsets of } P \text{ that separates points}$$

$$\mathcal{F} = \text{a set of upsets of } P \text{ that separates points}$$

$$R \subseteq \mathcal{I} \times \mathcal{F} \text{ is given by } I\,R\,F \Leftrightarrow I \cap F \neq \emptyset$$

Theorem Let $G(\mathcal{I}, \mathcal{F})$ be the Galois closed subsets given by the polarity from R. Then $P \leq G(\mathcal{I}, \mathcal{F})$ is a completion.

Template for Completions

Fact A relation $R \subseteq X \times Y$ gives a Galois connection between the power sets of X and Y. This is called a polarity.

Suppose

$$\mathcal{I} = \text{a set of downsets of } P \text{ that separates points}$$

$$\mathcal{F} = \text{a set of upsets of } P \text{ that separates points}$$

$$R \subseteq \mathcal{I} \times \mathcal{F} \text{ is given by } I \, R \, F \Leftrightarrow I \cap F \neq \emptyset$$

Theorem Let $G(\mathcal{I}, \mathcal{F})$ be the Galois closed subsets given by the polarity from R. Then $P \leq G(\mathcal{I}, \mathcal{F})$ is a completion.

Note Each of the above completions arises from this template with a suitable choice of $\mathcal{I}, \mathcal{F}$.

Characterization of $G(\mathcal{I}, \mathcal{F})$

Theorem $P \leq G(\mathcal{I}, \mathcal{F})$ is the unique completion where

- each element of $G(\mathcal{I}, \mathcal{F})$ is a join of meets and a meet of joins of elements of P.
- If $S, T \subseteq P$, then $\bigwedge S \leq \bigvee T$ iff $I \cap F \neq \emptyset$ for each $S \subseteq F \in \mathcal{F}$ and $T \subseteq I \in \mathcal{I}$.

Note This says $G(\mathcal{I}, \mathcal{F})$ preserves exactly those joins and meets under which each $I \in \mathcal{I}$, $F \in \mathcal{F}$ are closed.

Note This allows tailor-made completions.

Question Can you see Stone duality hiding in here?

Extending Operations

If $P \leq C$ and $f : P \to P$ is order preserving let $f^-, f^+ : C \to C$ be

$$f^-(c) = \bigvee\{f(p) : p \leq c\}$$
$$f^+(c) = \bigwedge\{f(p) : c \leq c\}$$

If $f : P^2 \to P$ preserves order in the 1^{st} coord, reverses it in 2^{nd}

$$f^-(c,d) = \bigvee\{f(p,q) : p \leq c, d \leq q\}$$
$$f^+(c,d) = \bigwedge\{f(p,q) : c \leq p, q \leq d\}$$

Defn f is monotone if it preserves or reverses order in each coord.

Note Any monotone f has extensions f^-, f^+ to C.

Extending Operations

The f^-, f^+ extensions are defined for any completion $P \leq C$, but are intended for completions that are join or meet dense.

For canonical completions, each $c \in C$ is both a join of meets and a meet of joins of elements of P. Define $f^\sigma, f^\pi : C \to C$ by

$$f^\sigma(c) = \bigvee\{\bigwedge f[S] : \bigwedge S \leq c\}$$
$$f^\pi(c) = \bigwedge\{\bigvee f[S] : c \leq \bigvee S\}$$

These definitions extend to arbitrary monotone operations.

Preserving Identities

Question When will a variety V of lattices with additional monotone operations be closed under a certain type of completion?

Rough overview

MacNeille completions work well in some important cases such as Boolean algebras, and the variety of all Heyting algebras, but usually behave poorly. Even distributive lattices are not closed under MacNeille completions.

Ideal completions work whenever the operations of V are order preserving. Otherwise, they work poorly. Boolean algebras are not closed under ideal completions.

Canonical completions ...

Preserving Identities

Some deeper results ...

Sahlqvist Theorems

These are results that give sufficient syntactic conditions for an identity $p \approx q$ to be preserved under completions.

- For BAO's and canonical comp's (Sahlqvist '75)
- For BAO's and canonical comp's (Jónsson '94)
- For BA's with conjugated op's and MacNeille comp's (G+V)
- For DMA's with canonical comp's (G+N+V)

Note Sahlqvist's original was via Kripke frames, Jónsson's is order-theoretic, hence portable. Givant & Venema and Gehrke, Nagahashi & Venema used Jónsson's method.

Preserving Identities

Theorem If K is a class of lattices with monotone op's that is closed under ultraproducts and canonical comp's, then the variety $V(K)$ it generates is closed under canonical comp's.

Proof Canonical comp's always preserve subalgebras and homomorphic images. For products, any $L = \prod L_i$ has a Boolean product representation $L \leq \prod U_x$ where the U_x are the ultraproducts of the L_i. Then the canonical comp of L is the full product $\prod U_x$.

Corollary Any variety generated by a single finite lattice with monotone op's is closed under canonical comp's.

Preserving Identities

Theorem If the variety V is closed under MacNeille completions then it is closed under canonical completions.

Proof For any L the canonical completion of L can be shown to be a subalgebra of the MacNeille completion of a suitably saturated ultrapower of L.

Preserving Identities

Some limiting results ...

- There are varieties that do not admit any completion at all. Examples are *l*-groups, Gödel-Lob algberas, modular OL's (proofs are trivial, cute, and very deep).
- There is a variety of modal algebras that is not closed under MacNeille completions using either the f^- or f^+ extension, but is closed using some other (non-constructive) extension. This result is equivalent to some weak version of choice.
- There is a variety of Heyting algebras that is not closed under MacNeille comp's yet admits a regular completion (it is $V(\mathbf{3})$).

Concluding Remarks

The last 15 years have seen a movement from isolated results about completions to the start of a systematic study. We have seen

- Canonical comp's move to the general lattice setting
- A Sahlqvist type result for MacNeille comp's
- Closed under MacNeille $\Rightarrow$ closed under canonical
- General framework to study comp's $G(\mathcal{I}, \mathcal{F})$.

Some questions that may help to continue this development follow.

Open Problems

Questions about canonical completions for general structures.

- Which varieties of lattices are closed under canonical comp's?
- Develop a Sahlqvist theory for general canonical comp's
- Develop a Kripke-type semantics and correspondence theory

Open Problems

Questions about canonical completions for general structures.

- Which varieties of lattices are closed under canonical comp's?
- Develop a Sahlqvist theory for general canonical comp's
- Develop a Kripke-type semantics and correspondence theory

Questions about regular completions.

- Do any varieties of lattices admit a regular completion?
- Does the variety of linear Heyting algebras admit a regular completion?

Open Problems

General questions

- Develop the general theory of $G(\mathcal{I}, \mathcal{F})$
- Is it decidible when the variety generated by a single finite structure is closed under MacNeille completions?
- What is the connection between comp's and ultraproducts?

Open Problems

General questions

- Develop the general theory of $G(\mathcal{I}, \mathcal{F})$
- Is it decidible when the variety generated by a single finite structure is closed under MacNeille completions?
- What is the connection between comp's and ultraproducts?

Old problems (over 50 years)

- Does the variety of $\textsc{oml}$'s admit a completion?
- If a lattice is complete and every element has exactly one complement, must it be Boolean?

Many thanks to the organizers.

Thank you for listening.

Papers at www.math.nmsu.edu/$\sim$jharding