Subalgebras of Orthomodular Lattices

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Outline

1. Preliminaries
2. Main Result
3. Why I am interested in the main result
4. Remarks on the proof
5. Further results
6. Open problems

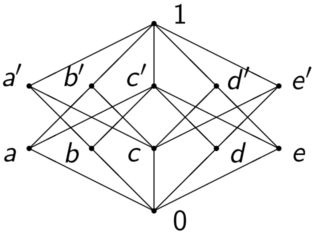
Preliminaries

Definition $(L, \wedge, \vee, ', 0, 1)$ is an orthomodular lattice (OML) if

1. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice
2. $x \leq y \Rightarrow y' \leq x'$
3. $x'' = x$
4. $x \wedge x' = 0$ and $x \vee x' = 1$
5. $x \leq y \Rightarrow x \vee (x' \wedge y) = y$

Preliminaries – Examples

1. Any Boolean algebra (BA) is an OML.

2. 

Note Like all OMLs, this one is formed by *gluing* together BAs. The devil is in the way the glueing is done.

Preliminaries – Examples

3. *Proj* $\mathcal{H}$ for a Hilbert space $\mathcal{H}$.

4. *Proj* $\mathcal{M}$ for any von Neumann (VN) algebra $\mathcal{M}$.

These are key in quantum mechanics and motivated OMLS.

Observables	$\equiv$	*Borel* $\mathbb{R} \to$ *Proj* $\mathcal{H}$	(Spectral)
States	$\equiv$	*Proj* $\mathcal{H} \to [0,1]$	(Gleason)
Unitary+antiunitary	$\equiv$	auto's of *Proj* $\mathcal{H}$	(Wigner)

Note In *Proj* $\mathcal{H}$ the maximal Boolean subalgebras (blocks) correspond to orthonormal bases of $\mathcal{H}$. It is how they overlap that gives the structure of $\mathcal{H}$ and *Proj* $\mathcal{H}$.

Main Result

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Theorem For OMLs L and M and a poset isomorphism

$$f : BSub(L) \to BSub(M)$$

there is an OML-isomorphism $F : L \to M$ with $f(S) = F[S]$ for each Boolean subalgebra S of L. Further, this map F is unique provided L has no blocks with four elements.

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Corollary L is determined up to isomorphism by $BSub(L)$.

Why?

In the 1990's, Isham and Butterfield introduced a topos approach to quantum mechanics.

Quantum notions were replaced with sheaf-theoretic versions of "more classical" counterparts over some base space X.

Example They show the Kochen-Specker theorem is equivalent to a certain sheaf having no global element.

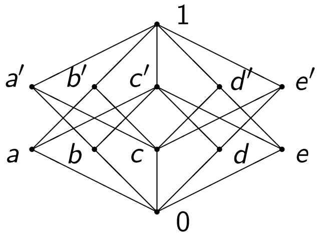
Why?

Isham and Butterfield creat the base space X as follows:

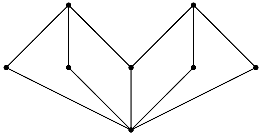
- Start with a Hilbert space $\mathcal{H}$ or VN-algebra $\mathcal{M}$
- Make OML $L = \mathit{Proj}\ \mathcal{H}$ or $\mathit{Proj}\ \mathcal{M}$
- Form the frame of downsets of $BSub(L)$
- $X =$ corresponding space

Actually, they just work with the frame.

Remarks on Proof – Example



L



$BSub(L)$

Maximal elements of $BSub(L)$ = Blocks of L

Atoms of $BSub(L) = \{0, a, a', 1\}, \{0, b, b', 1\}, \ldots, \{0, e, e', 1\}$

Remarks on the Proof

We can easily prove some interesting special cases.

Proposition The subalgebras $\{0, x, x', 1\}$ where $x \in L - \{0, 1\}$ are exactly the atoms of $BSub(L)$.

Corollary $|L| = 2n + 2$ where $n =$ number of atoms of $BSub(L)$.

Corollary A finite BA B is determined by $Sub(B)$.

Remarks on the Proof

Proposition For $x, y \in L$ TFAE

1. x is comparable to one of y, y'
2. $\{0, x, x', 1\} \vee \{0, y, y', 1\}$ exists & has height ≤ 2

Proposition Let $x \in L - \{0, 1\}$ and set $a = \{0, x, x', 1\}$. TFAE

1. At least one of x, x' is an atom of L.
2. b and atom of $BSub(L)$ & $a \vee b$ exists $\Rightarrow$ height $a \vee b \leq 2$.

As we can determine the atoms of L and when $x \leq y'$ for atoms ...

Corollary If L is a finite OML it is determined by $BSub(L)$.

Remarks on the Proof

Lets describe the general case ...

- Use Sachs' solution for the general case for BAs (1962).
- This provides solutions for blocks of L
- Show we can piece these together to solve things for L.

Remarks on the Proof

Let $BSub(L) \xrightarrow{f} BSub(M)$, Boolean $B \leq L$ and $C = f(B)$

$$\begin{array}{ccc} [0, B] & \xrightarrow{f_B} & [0, C] \\ || & & || \\ Sub(B) & \longrightarrow & Sub(C) \end{array} \qquad \overset{Sachs}{\Rightarrow} \qquad B \xrightarrow{F_B} C$$

Show F_B's agree on their overlap (a bit tricky).

Further Results

Theorem Our main theorem with $BSub(L)$ replaced by $Sub(L)$.

Theorem A duality between the category of OMLs having no 4-element blocks with onto homomorphism as morphisms, and a subcategory of the category of algebraic lattices with morphisms preserving arbitrary meets and up-directed joins.

Note This sounds nice, I doubt it is much use.

Further results

Definition For a von Neumann algebra $\mathcal{M}$ let $AbSub\ \mathcal{M}$ be its poset of abelian VN-subalgebras.

Theorem For VN-algebras $\mathcal{M}$, $\mathcal{N}$ without type I_2 summand and poset isomorphism

$$f : AbSub\ \mathcal{M} \to AbSub\ \mathcal{N}$$

there is a unique Jordan iso $F : \mathcal{M} \to \mathcal{N}$ with $f(S) = F[S]$ for each abelian subalgebra S of $\mathcal{M}$.

Note This is the most we can ask as there exists $\mathcal{M} \not\approx \mathcal{M}^{op}$.

Open Problems

1. Describe the posets that occur as $BSub(L)$ for some OML L.

2. Is a C^* algebra A determined up to Jordan iso by $AbSub(A)$? (Yes if it is abelian)

3. Is $AbSub\ \mathcal{M}$ linked to orientations of the Jordan part of $\mathcal{M}$?

Thank you for listening.

Papers at www.math.nmsu.edu/$\sim$jharding