Proximities — A Survey

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Introduction

This is some work I've done with Guram and Nick Bezhanishvili about proximities. All was with Guram, about a third with Nick.

Proximities are an old notion in topology, dating at least to Freudenthal (≈ 1940). They arise in describing compactifications, and in de Vries duality for compact Hausdorff spaces.

Our results (1) extend de Vries duality, (2) extend the coalgebraic treatment of modal logic, (3) describe stable compactifications.

Much of the talk will discuss background, just a taste of details.

Topology and lattice theory

Given a topological space X, its open sets $\mathfrak{O}X$ form a complete lattice under set inclusion where

$$\bigvee A_i = \bigcup A_i \qquad \text{(least upper bound)}$$

$$\bigwedge A_i = \mathrm{I} \bigcap A_i \qquad \text{(greatest lower bound)}$$

Call least upper bounds joins, greatest lower bounds meets. Note that finite meets in $\mathfrak{O}X$ are given by intersections since a finite intersection of open sets is open.

Topology and lattice theory

Definition: A complete lattice is a frame if it satisfies

$$a \wedge \bigvee b_i = \bigvee a \wedge b_i$$

Corollary: $\mathfrak{O}X$ is a frame.

Topological spaces and frames are not the same thing, but there is an equivalence between "nice" spaces and "nice" frames. For nice spaces, the frame of opens determines everything about the space. Every Hausdorff space is nice, and many others are too.

Topology and lattice theory

Little Quiz: Consider the frame L below.

Is L the frame of opens of a Hausdorff space?

Is L the frame of opens of a compact space?

Topology and lattice theory

A is regular open if $A = \mathrm{IC}A$.

Example: $(0,1)$ is regular open, $(0,1) \cup (1,2)$ is not.

The regular open sets $\mathfrak{R}X$ are a complete Boolean algebra where

$$\begin{aligned} \bigvee A_i &= \mathrm{IC}\bigcup A_i \\ \bigwedge A_i &= \mathrm{I}\bigcap A_i \\ \neg A &= \mathrm{I}(X - A) \end{aligned}$$

For "nice" (sober) spaces, the open sets determine everything. But the regular opens do not, we need some extra information.

Basics of de Vries duality

For a space X define $A \prec B$ if $\mathrm{C}A \subseteq B$.

Theorem: If X is a compact Hausdorff space, then for $\prec$ on $\mathfrak{R}X$

1. $1 \prec 1$.
2. $a \prec b$ implies $a \leq b$.
3. $a \leq b \prec c \leq d$ implies $a \prec d$.
4. $a \prec b, c$ implies $a \prec b \wedge c$.
5. $a \prec b$ implies $\neg b \prec \neg a$.
6. $a \prec b$ implies there exists c with $a \prec c \prec b$.
7. $a = \bigvee \{b : b \prec a\}$.

Definition: A complete Boolean algebra with relation $\prec$ satisfying the above conditions is a de Vries algebra.

Basics of de Vries duality

Definition: $f : (B, \prec) \to (B', \prec)$ is a de Vries morphism if

1. f preserves bounds and finite meets
2. $a \prec b \Rightarrow \neg f(\neg a) \prec f(b)$
3. $f(a) = \bigvee\{f(b) : b \prec a\}$

The de Vries algebras and morphisms form a category, but under an unusual definition of composition.

Basics of de Vries duality

Theorem: The category of de Vries algebras is dually equivalent to the category of compact Hausdorff spaces.

- Send a space X to $(\mathfrak{R}X, \prec)$
- Send $(B, \prec)$ to its space of maximal round filters (ends).

A filter F is round if $b \in F \Rightarrow \exists a \in F, a \prec b$.

Basics of de Vries duality

Example: Let $X = \mathbb{N} \cup \{\infty\}$, the 1-point compactification of $\mathbb{N}$

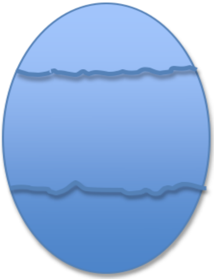
$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 1 & 2 & 3 & & \infty \end{array}$$

Opens $=$ all $A \subseteq \mathbb{N}$ and all cofinite A containing ∞.

Regular opens $=$ non-cofinite $A \subseteq \mathbb{N}$ and cofinite A containing ∞.

For regular opens, $A \prec B$ iff $A \subseteq B$ and A finite or B cofinite.

So the de Vries algebra of $X = \mathbb{N} \cup \{\infty\}$ looks as follows.



The maximal round filters of this de Vries algebra are as follows.

$\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$

$\uparrow\{0\} \quad \uparrow\{1\} \quad \uparrow\{2\} \quad \quad$ top part

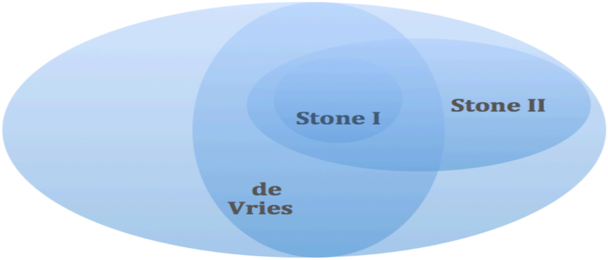
First result — Extending de Vries duality

Several well-known dualities ...

Stone I 0-dim compact Hausdorff spaces $\leftrightarrow$ Boolean algebras

Stone II spectral spaces (spectra of rings) $\leftrightarrow$ distributive lattices

de Vries compact Hausdorff spaces $\leftrightarrow$ de Vries algebras



First result — Extending de Vries duality

Definition X is stably compact if it is compact, locally compact, sober ("nice"), and A, B compact saturated $\Rightarrow A \cap B$ is compact.

Note This generalizes compact Hausdorff and spectral spaces.

Definition: A proximity frame is a frame L with relation $\prec$ where

1. $0 \prec 0, 1 \prec 1$.
2. $a \prec b$ implies $a \leq b$.
3. $a \leq b \prec c \leq d$ implies $a \prec d$.
4. $a \prec b, c$ implies $a \prec b \wedge c$.
5. $a, b \prec c$ implies $a \vee b \prec c$.
6. $a \prec b$ implies there exists c with $a \prec c \prec b$.
7. $a = \bigvee \{b : b \prec a\}$.

First result — Extending de Vries duality

Thm There is a duality stably compact spaces $\leftrightarrow$ proximity frames.

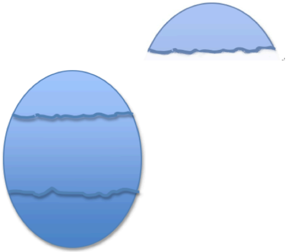
- Send stably compact X to $(\mathfrak{O}X, \prec)$ where $A \prec B$ iff $\complement A \subseteq B$.
- Send proximity frame $(L, \prec)$ to its space prime round filters.

However There are two big drawbacks. This doesn't generalize de Vries duality as it uses $\mathfrak{O}X$ not $\mathfrak{R}X$, and the category of proximity frames is odd (isomorphisms are strange).

First result — Extending de Vries duality

Example: Let $X = \mathbb{N} \cup \{\infty\}$, the 1-point compactification of $\mathbb{N}$

Opens $=$ all $A \subseteq \mathbb{N}$ and all cofinite A containing ∞.



$A \prec B$ iff $A \subseteq B$ and either A finite or B cofinite containing ∞.

First result — Extending de Vries duality

We can fix these defects. Use $\to$ for Heyting implication.

Definition For a proximity frame L and $a \in L$ set

$$\begin{aligned} ka &= \bigwedge\{b : a \prec b\} \\ ja &= \bigwedge\{(a \to kb) \to kb : b \in L\} \end{aligned}$$

Call L regular if $ja = a$ for all $a \in L$.

Theorem The fixed points of j form a regular proximity frame.

First result — Extending de Vries duality

Theorem de Vries duality extends to a duality between stably compact spaces and regular proximity frames.

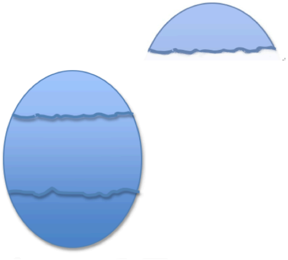
- Send X to the j-fixed points of $(\mathfrak{O}X, \prec)$.
- Send $(L, \prec)$ to its space of prime round filters.

Note A stably compact space has an associated "patch" topology. The j-fixed points of $\mathfrak{O}X$ are those A that are regular open in the sense $A = \mathrm{I}\mathrm{C}A$ where I is in the given topology, C in the patch.

First result — Extending de Vries duality

Example Consider again $X = \mathbb{N} \cup \{\infty\}$.

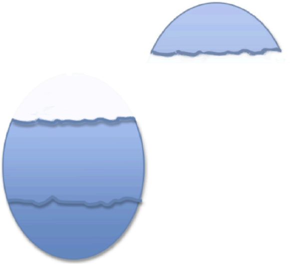
Recall the proximity frame of open sets $\mathfrak{O}X$.



First result — Extending de Vries duality

Example Consider again $X = \mathbb{N} \cup \{\infty\}$.

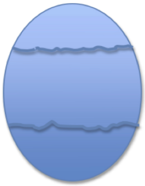
Now the j-fixed points of $\mathfrak{O}X$.



First result — Extending de Vries duality

Example Consider again $X = \mathbb{N} \cup \{\infty\}$.

Redrawing this, we arrive at our de Vries algebra from before.



Second result — compactifications

A (classical) compactification of a Hausdorff space X is a compact Hausdorff space Y having X as a dense subspace.

The compactifications of X are quasiordered by setting $Y \leq Y'$ if there is a continuous surjection $Y' \to Y$ that is the identity on X.

The Stone-Cech compactification is the largest, there may or may not be a smallest.

Second result — compactifications

One of the nice parts of de Vries duality is the following version of a famous result of Smyrnov.

Theorem The poset of compactifications of a Hausdorff space X is isomorphic to the poset of de Vries proximities on $\mathfrak{R}X$.

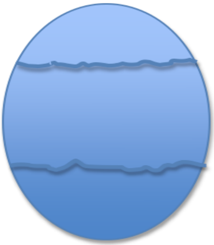
We obtain an analogous spin on a result of Smyth.

Theorem The poset of stable compactifications of any T_0 space is isomorphic to the poset of proximities on $\mathfrak{O}X$.

We hope the recent introduction of j will add further insight here.

Second result — compactifications

Example Consider classical compactifications of $X = \mathbb{N}$.



Classical compactifications are given by de Vries proximities.

- $\leq$ is the largest (gives Stone-Cech).
- $A \prec B$ iff A finite or B cofinite smallest (one-point).

Third result — coalgebras & modal logic

Modal logic adds an extra connective $\diamond$ for "possibly" to our usual connectives "and", "or", "not".

Modal logic is treated algebraically via modal algebras. These are Boolean algebras with an additional unary operator (B, f) where

- $f(0) = 0$
- $f(a_1 \vee a_2) = fa_1 \vee fa_2$

The first says "possibly false" is false.
The second "possibly a_1 or a_2" $=$ "possibly a_1" or "possibly a_2".

Third result — coalgebras & modal logic

Kripke semantics treats modal logic via the Stone space X of B. This turns f into a binary relation on X where xRy iff $f[x] \subseteq y$.

Any binary relation R on a set X can be treated as a map

$$R : X \to \mathcal{P}X \text{ (power set)} \quad x \rightsquigarrow R[\{x\}]$$

Theorem A relation R on X comes from a modal operator f iff

$$R : X \to \mathcal{V}X \text{ (Vietoris space)} \quad \text{is continuous.}$$

Third result — coalgebras & modal logic

So modal algebras correspond to 0-dimensional compact Hausdorff spaces X with a binary relation R with $R : X \to \mathcal{V}X$ continuous.

This can be extended to a categorical duality between modal algebras and so-called modal spaces.

In theoretical computer science, they view these $R : X \to \mathcal{V}X$ as coalgebras for the Vietoris functor on 0-dimensional compact Hausdorff spaces.

Third result — coalgebras & modal logic

Extend things by considering the Vietoris functor on all compact Hausdorff spaces.

Definition A de Vries modal space is a compact Hausdorff space X with binary relation R where

$$R : X \to \mathcal{V}X \text{ (Vietoris space)} \quad \text{is continuous.}$$

These de Vries modal spaces are coalgebras for the Vietoris functor on compact Hausdorff spaces.

Third result — coalgebras & modal logic

We then seek the algebraic counterpart of modal de Vries spaces.

Definition A modal de Vries algebra is a de Vries algebra $(B, \prec)$ with unary operation $f : B \to B$ satisfying

1. $f0 = 0$
2. $a_1 \prec b_1, a_2 \prec b_2 \Rightarrow f(a_1 \vee a_2) \prec fb_1 \vee fb_2$.

Note This becomes the usual additivity for a modal operator if you replace $\prec$ by $\leq$. We call this de Vries additivity.

Third result — coalgebras & modal logic

After defining appropriate morphisms ...

Theorem The category of modal de Vries algebras is dually equivalent to the category of modal de Vries spaces.

Note This extends both de Vries duality and the usual duality used in modal logic between modal logics and modal spaces.

We have the start of a nice theory for modal de Vries algebras, but lots could still be done.

Some achievable research projects

There are many open lines where one can expect results ...

There are special kinds of modal algebras, such as closure algebras. One can say much more about these, and their associated relations. Is there a nice theory of de Vries closure algebras?

Is there a version of modal logic for stably compact spaces? This would give a duality that encompasses all the ones we have discussed. It would also encompass distributive modal algebras.

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding