CANONICAL EXTENSIONS, FREE COMPLETELY DISTRIBUTIVE LATTICES, AND COMPLETE RETRACTS

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A recent paper of Morton and van Alten [7] shows, among other things, that the canonical extension D^σ of a bounded distributive lattice D is the free completely distributive lattice generated by D. This means D^σ is completely generated by D and for any completely distributive lattice C and any bounded lattice homomorphism $h : D \to C$, there is a complete lattice homomorphism $h^* : D^\sigma \to C$ extending h. The purpose of this note is to provide a short alternate proof of this result, and to show its utility in easily establishing results about completely distributive lattices, canonical extensions, and complete retractions, both known and new. We assume familiarity with basics of canonical extensions as given in the first 8 pages of [4].

Theorem 1. *For a bounded distributive lattice D, the canonical extension D^σ is the free completely distributive lattice over D.*

Proof. That D has a free completely distributive extension F follows from a standard argument using the facts that every bounded distributive lattice can be embedded into a completely distributive one (its canonical extension), and that the cardinality of a bounded distributive lattice completely generated by D is at most $2^{2^{|D|}}$. We show that the free completely distributive extension is the canonical extension. To do so, we must show density and compactness [4, Def. 2.5]. Density holds since F is completely distributive and completely generated by D. For compactness, let $A, B \subseteq D$ with $\bigwedge A \leq \bigvee B$ in F. With $f : F \to D^\sigma$ the complete homomorphism extending the identity on D, $\bigwedge f[A] \leq \bigvee f[B]$. Compactness of the canonical extension D^σ gives finite subsets $A' \subseteq A$ and $B' \subseteq B$ with $\bigwedge A' \leq \bigvee B'$. $\square$

Using a standard fact about reflectors [1, Thm. 2, p. 28], the following is immediate from Theorem .

Corollary 2. *The canonical extension σ is a reflector from the category* DL *of bounded distributive lattices and bounded lattice homomorphisms to its (non-full) subcategory* CD *of completely distributive lattices and complete homomorphisms.*

As our second corollary, we obtain the following result of Markowsky [6].

Corollary 3. *The free completely distributive lattice over a set X is the canonical extension of the free bounded distributive lattice over X, hence is isomorphic to the lattice* $\mathsf{Up}(\wp\mathsf{X})$ *of upsets of the powerset of X.*

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Proof. Let D be the free bounded distributive lattice over the set X. Since D^σ is the free completely distributive lattice over D, it is the free completely distributive lattice over X. It is well known [1, p. 90] that as a poset, the Priestley space of D is isomorphic to the powerset of X. Since D^σ is realized as the upsets of the Priestley space of D, the result follows. ☐

Suppose R and C are complete lattices and R is a bounded sublattice of C, i.e. the identical embedding $\iota : R \to C$ is a bounded lattice homomorphism. Then R is a *complete retract* of C if there is a complete homomorphism $r : C \to R$ with $r \circ \iota = 1_R$. In this case, r is called a *complete retraction*. The following is immediate from Theorem .

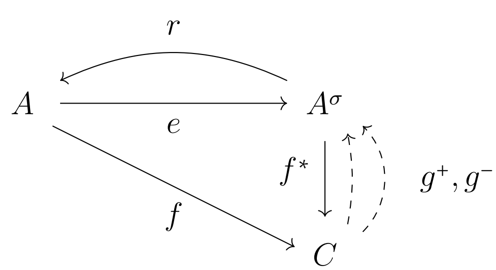
Corollary 4. *A completely distributive lattice is a complete retract of its canonical extension.*

We use the term *Raney lattice* for a lattice that is isomorphic to the upsets of a poset. There are many characterizations of Raney lattices. For example, they are exactly the completely distributive algebraic lattices. Other authors have referred to them as superalgebraic lattices or perfect lattices. Since the canonical extension D^σ of a bounded distributive lattice D is realized as the upsets of its Priestley space, we obtain the following result of Raney [8].

Corollary 5. *Completely distributive lattices are the complete retracts of Raney lattices.*

Theorem 6. *If A is a bounded sublattice of C, both are completely distributive, and A completely generates C, then A is a complete retract of C.*

Proof. Let $e : A \to A^\sigma$ and $f : A \to C$ be identical embeddings. Since A is completely distributive, freeness of A^σ provides there is a complete retraction $r : A^\sigma \to A$. Freeness of A^σ also provides a complete lattice homomorphism $f^* : A^\sigma \to C$ with $f^* \circ e = f$.



Since C is completely generated by the image of f and f^* extends f and is complete, it follows that f^* is onto. Since f^* is complete, it has both a left adjoint g^- and right adjoint g^+, one of which preserves joins, and the other meets.

Claim: $r \circ g^- = r \circ g^+$.

Proof of claim. Let $c \in C$. Since f^* is onto, $g^-(c)$ is the least element mapped by f^* to c and $g^+(c)$ is the largest. Therefore, $g^-(c) \leq g^+(c)$, and so $r(g^-(c)) \leq r(g^+(c))$. For the converse, since each element of A^σ is the meet of the open elements of A^σ above it, open elements of A^σ are exactly the joins of ideals of A, and $g^-(c)$ is the meet of all elements of A^σ that are mapped by f^* above c, we have

$$g^-(c) = \bigwedge\{\bigvee e(I) \mid I \text{ is an ideal of } A \text{ and } c \leq f^*(\bigvee e(I))\}.$$

Since every element of A^σ is a join of closed elements, we obtain a dual expression for $g^+(c)$. Using the facts that r and f^* are complete, that $r \circ e = 1_A$, and that $f^* \circ e = f$, we have

$$r(g^-(c)) = \bigwedge\{\bigvee I \mid I \text{ is an ideal of } A \text{ and } c \le \bigvee f(I)\},$$
$$r(g^+(c)) = \bigvee\{\bigwedge F \mid F \text{ is a filter of } A \text{ and } \bigwedge f(F) \le c\}.$$

Suppose that I is an ideal of A and F a filter of A such that $\wedge f(F) \le c \le \vee f(I)$. By general principles, $f(\wedge F) \le \wedge f(F)$ and $\vee f(I) \le f(\vee I)$. So $f(\wedge F) \le f(\vee I)$. Since f is a lattice embedding, it follows that $\wedge F \le \vee I$. So $r(g^+(c)) \le r(g^-(c))$, yielding equality. □

Set $s = r \circ g^- = r \circ g^+$. Since r is complete, g^- preserves joins, and g^+ preserves meets, it follows that s is complete. Let $a \in A$. Since $f^* \circ e = f$, we have

$$g^-(f(a)) \le e(a) \le g^+(f(a)).$$

Applying r to the terms in this expression yields $s \circ f = 1_A$. So $s : C \to A$ is a complete retraction. □

Example 7. The 2-element chain is a bounded sublattice of the real unit interval but is not a complete retract. So the complete generation condition in Theorem cannot be removed.

We now use Theorem to establish a result of Johnstone [5, p. 293] that strengthens Corollary . We assume the reader is familiar with the basics of frames [5]. We use the fact that each completely distributive lattice is a spatial frame, which follows from Raney's result [9] that every completely distributive lattice can be completely embedded into a product of complete chains.

Corollary 8. *If C is a completely distributive lattice, there is a frame embedding $e : C \to R$ into a Raney lattice and a complete retraction $r : R \to C$ with $r \circ e = 1_C$.*

Proof. Let R be the upset lattice $\mathsf{Up}(\mathrm{pt}C)$ of the poset of points of C. Since C is spatial, there is a frame embedding $e : C \to R$ given by $e(a) = \{p \mid p(a) = 1\}$. For each point p we have $\bigcap\{e(a) \mid p(a) = 1\} = {\uparrow}p$ is the upset of $\mathrm{pt}C$ generated by p. It follows that the image of e completely generates R. The result then follows from Theorem . □

To further consider complete retractions, recall that Raney duality between the category of Raney lattices and complete lattice homomorphisms and the category of posets and order-preserving maps takes a poset to its lattice of upsets and an order-preserving map to its inverse image. Under this duality order-embeddings correspond to onto complete homomorphisms and onto order-preserving maps to one-to-one complete homomorphisms. The restriction of Raney duality to the Boolean setting is the duality between complete atomic Boolean algebras and sets known as Tarski duality. Raney duality is largely folklore (see [2] for details).

Proposition 9. *Suppose B is a complete sublattice of a Raney lattice C. If B and C are Boolean, or are chains, then B is a complete retract of C.*

Proof. Raney duality provides that a complete sublattice of a Raney lattice is Raney. Let X and Y be the Raney duals of B and C, respectively. Then there is an onto order-preserving map $f : Y \to X$. In the case that B and C are Boolean, X and Y are anti-chains. In the case when B and C are chains, X and Y are chains. In either case there is an order-embedding $g : X \to Y$ with $f \circ g = 1_X$. Raney duality provides a complete retraction from C to B. □

Corollary 10. *Let A be completely distributive, C a Raney lattice, and A a bounded sublattice of C. If A and C are Boolean, or are chains, then A is a complete retract of C.*

Proof. Let B be the complete sublattice of C generated by A. Then B is a complete retract of C by Proposition and A is a complete retract of B by Theorem . □

The following question appears to be open even in the finite setting.

Problem. *Characterize when a complete lattice A that is a bounded sublattice of a completely distributive lattice C is a complete retract.*

Remark 11. Using Birkhoff duality between finite distributive lattices and finite posets, conditions for a sublattice A of a finite distributive lattice D to be a retract is equivalent to the following problem for posets. Given an onto order-preserving map $f : P \to Q$ between finite posets, determine when there is an order-embedding $s : Q \to P$ with $f \circ s = 1_Q$. A related problem for finite posets, determining when an order-embedding $s : Q \to P$ admits a retraction $f : P \to Q$, has been considered [3].

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