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Raney Algebras and Duality for T_0-Spaces

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Abstract

In this note we adapt the treatment of topological spaces via Kuratowski closure and interior operators on powersets to the setting of T_0-spaces. A Raney lattice is a complete completely distributive lattice that is generated by its completely join prime elements. A Raney algebra is a Raney lattice with an interior operator whose fixpoints completely generate the lattice. It is shown that there is a dual adjunction between the category of topological spaces and the category of Raney algebras that restricts to a dual equivalence between T_0-spaces and Raney algebras. The underlying idea is to take the lattice of upsets of the specialization order with the restriction of the interior operator of a space as the Raney algebra associated to a topological space. Further properties of topological spaces are explored in the dual setting of Raney algebras. Spaces that are T_1 correspond to Raney algebras whose underlying lattices are Boolean, and Alexandroff T_0-spaces correspond to Raney algebras whose interior operator is the identity. Algebraic description of sober spaces results in algebraic considerations that lead to a generalization of sober that lies strictly between T_0 and sober.

Keywords Topological space · T_0-space · T_1-space · Alexandroff space · Sober space · Closure algebra · Interior algebra

Mathematics Subject Classification 54D10 · 06D22 · 06E25 · 06D10

1 Introduction

Kuratowski [7] gave an alternate means to define a topology on a set X through a *closure operator* on the powerset, an operator $\Diamond$ that satisfies $a \leq \Diamond a$, $\Diamond\Diamond a \leq \Diamond a$, $\Diamond 0 = 0$, and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. Abstracting from such operators on powersets to Boolean algebras gave rise to the *closure algebras* of McKinsey and Tarski [9], who among other things showed that closure algebras are algebraic models of the modal logic **S4**. Because of this, closure algebras

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are often referred to as **S4**-algebras in the modal logic literature. Passing to complements, one obtains an *interior operator* on the powerset, an operator $\Box$ that satisfies $\Box a \le a$, $\Box a \le \Box\Box a$, $\Box 1 = 1$, and $\Box(a \wedge b) = \Box a \wedge \Box b$. The corresponding algebras were studied by Rasiowa and Sikorski [12] under the name of *topological Boolean algebras* and by Blok [3] under the name of *interior algebras*.

Tarski [13] showed that powerset Boolean algebras are characterized up to isomorphism as complete and atomic Boolean algebras. It follows from this and Kuratowski's result mentioned above that topological spaces are characterized as interior algebras $(B, \Box)$ whose underlying Boolean algebra is complete and atomic. Functions $f : X \to Y$ between sets correspond, via inverse image $h = f^{-1}$, to Boolean algebra homomorphisms $h : \wp(Y) \to \wp(X)$ between their powersets that preserve arbitrary joins and meets. If X and Y are topological spaces, then f is continuous iff the preimage of the interior of a set is always contained in the interior of its preimage (see, e.g., [12, Sec. III.3]). These results have convenient categorical formulation.

Let Top be the category of topological spaces and continuous maps and MT the category whose objects are complete and atomic interior algebras and whose morphisms are complete Boolean homomorphisms h satisfying $h(\Box a) \le \Box h(a)$. We chose to call this category MT after McKinsey and Tarski as they were one of the first to undertake an algebraic and order-theoretic study of topology. The results above are summarized as follows.

Theorem 1.1 *The categories* Top *and* MT *are dually equivalent.*

Some classes of topological spaces have simple order-theoretic characterization in terms of the category MT. For example, T_1-spaces are those where each singleton is the intersection of the opens that contain it, and T_0-spaces are those where each singleton is the intersection of open and closed sets that contain it. Let MT_0 be the full subcategory of MT whose objects are those $(B, \Box)$ where B is generated as a complete Boolean algebra by the $\Box$-fixpoints, and let MT_1 be the full subcategory of MT whose objects are those $(B, \Box)$ where B is generated as a complete lattice by the $\Box$-fixpoints. Then we have the following.

Theorem 1.2 *The dual equivalence between* Top *and* MT *restricts to a dual equivalence between* Top_0 *and* MT_0*, which further restricts to a dual equivalence between* Top_1 *and* MT_1.

$$
\begin{array}{ccccc}
\mathsf{Top}_1 & \hookrightarrow & \mathsf{Top}_0 & \hookrightarrow & \mathsf{Top} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathsf{MT}_1 & \hookrightarrow & \mathsf{MT}_0 & \hookrightarrow & \mathsf{MT}
\end{array}
$$

In this note we show that we can characterize T_0-spaces in a simpler signature that does not involve complementation. Our approach is to work with lattices of upsets instead of powerset algebras. We call these lattices *Raney lattices* due to his work in the subject (see, e.g., [11]). A *Raney algebra* is a pair $(R, \Box)$ where R is a Raney lattice, $\Box$ is an interior operator on R, and the $\Box$-fixpoints generate R as a complete lattice. Let RAlg be the category of Raney algebras and complete lattice homomorphisms h satisfying $h(\Box a) \le \Box h(a)$.

We show there is a contravariant adjunction between RAlg and Top which restricts to a dual equivalence between RAlg and Top_0. This leads to an adjunction between RAlg and MT that restricts to an equivalence between RAlg and MT_0. Let RAlg_1 be the full subcategory of RAlg dually equivalent to Top_1. We clearly have that RAlg_1 is equivalent to MT_1. But we have more, these two categories are literally equal—the Raney algebra and the interior algebra of a T_1-space are the same. In addition, we describe T_0-reflection, Alexandroff spaces, and sober spaces in the setting of our duality. We also provide a generalization of sober spaces suggested by our considerations.

2 Raney Lattices

Definition 2.1 A *quasi-ordered set* is a set Q with a reflexive and transitive relation $\leq$. For a quasi-ordered set Q and $S \subseteq Q$, let

$$\uparrow S = \{q \in Q \mid s \leq q \text{ for some } s \in S\} \text{ and } \downarrow S = \{q \in Q \mid q \leq s \text{ for some } s \in S\}.$$

We call S an *upset* if $S = \uparrow S$ and a *downset* if $\downarrow S = S$.

The collection $\mathsf{Up}(Q)$ of all upsets of Q forms a complete and completely distributive lattice under set inclusion with arbitrary joins and meets given by unions and intersections. For an order-preserving map $f\colon Q \to S$ the inverse map $f^{-1}\colon \mathsf{Up}(S) \to \mathsf{Up}(Q)$ is a complete lattice homomorphism. This defines a contravariant functor $\mathcal{U}\colon \mathsf{Qos} \to \mathsf{CD}$ where Qos is the category of quasi-ordered sets and order-preserving maps and CD is the category of complete and completely distributive lattices and complete lattice homomorphisms.

Definition 2.2 An element a of a complete lattice L is *completely join prime* if $a \leq \bigvee T$ implies $a \leq t$ for some $t \in T$.

For a complete and completely distributive lattice L, let X_L be the set of completely join prime elements of L. We consider X_L as a poset where the order is the dual of the restriction of the order on L. For a complete lattice homomorphism $h\colon L \to M$ let $h_*\colon M \to L$ be the left adjoint of h. Then $h_*\colon X_M \to X_L$ is a well-defined order-preserving map. This defines a contravariant functor $\mathcal{J}\colon \mathsf{CD} \to \mathsf{Pos}$ where Pos is the full subcategory of Qos consisting of posets.

Definition 2.3 Let $L \in \mathsf{CD}$. We call L a *Raney lattice* if each element of L is a join of completely join prime elements of L.

Raney lattices have a long history and can be characterized in many ways (see, e.g, [5, Thm. 10.29]). Among them is the next result, which we refer to as *Raney duality*, since it has origins in the work of Raney [11] where the object level of the correspondence was established (see also Balachandran [1] and Bruns [4]). The morphism level is not difficult to prove, but is hard to find in the literature.

Theorem 2.4 *The functors $\mathcal{U}$ and $\mathcal{J}$ give a contravariant adjunction between* Qos *and* CD*, which induces a dual equivalence between* Pos *and the full subcategory* Ran *of* CD *consisting of Raney lattices.*

$$\begin{array}{ccc}
\mathsf{Qos} & \underset{\mathcal{J}}{\overset{\mathcal{U}}{\rightleftarrows}} & \mathsf{CD} \\
\uparrow & & \uparrow \\
\mathsf{Pos} & \underset{\mathcal{J}}{\overset{\mathcal{U}}{\rightleftarrows}} & \mathsf{Ran}
\end{array}$$

The units $\varepsilon\colon 1_{\mathsf{Qos}} \to \mathcal{J}\mathcal{U}$ and $\eta\colon 1_{CD} \to \mathcal{U}\mathcal{J}$ of the contravariant adjunction are given by $\varepsilon_Q(x) = \uparrow x$ and $\eta_R(a) = \downarrow a \cap X_R = \{x \in X_R \mid x \leq a\}$.

In this paper we will attach Raney lattices to topological spaces, and will lift Raney duality to a duality for T_0-spaces. The fundamental ingredient is the specialization order of a topological space.

Definition 2.5 The *specialization order* of a topological space X is defined by $x \leq y$ iff x belongs to the closure of $\{y\}$.

It is well known that the specialization order is always a quasi-order, and it is a partial order exactly when X is a T_0-space. There is an adjunction between Top and Qos, taking a space X to the quasi-ordered set given by its specialization order, and taking a quasi-ordered set Q to the topological space whose opens are the upsets of Q. This adjunction restricts to an equivalence between Qos and the full subcategory of Top consisting of *Alexandroff spaces* (that is, spaces where arbitrary intersections of open sets are open). This equivalence restricts to an equivalence between Pos and the category of Alexandroff T_0-spaces.

3 Raney Algebras

In this section we enrich Raney lattices with an interior operator. To relate them to T_0-spaces we need an additional assumption that the fixpoints of the interior operator are dense.

Definition 3.1 Let L be a complete lattice. We say that $S \subseteq L$ is *join-meet dense* if each element of L is a join of meets of S; *meet-join dense* if each element of L is a meet of joins of S; and *dense* if S is both join-meet and meet-join dense in L.

If L is completely distributive, then it is easy to see that join-meet dense, meet-join dense, and dense are equivalent. From now on we will refer to this property as density.

Definition 3.2 We call a pair $(R, \Box)$ a *Raney algebra* if R is a Raney lattice, $\Box$ is an interior operator on R, and the fixpoints of $\Box$ are dense in R.

Recall (see, e.g., [6, Sec. II.1]) that a *frame* is a complete lattice in which finite meets distribute over arbitrary joins, and that a *subframe* of a frame is a subset that is closed under finite meets and arbitrary joins. Clearly each Raney lattice is a frame. For an interior operator $\Box$ on a Raney lattice R, we note that the fixpoints L of $\Box$ are exactly the image $\Box[R]$. It is well known and easy to see that L is a subframe of R. In fact, $\Box$ is right adjoint to the inclusion $L \hookrightarrow R$. Conversely, any subframe L of R gives rise to an interior operator $\Box$ on R as the right adjoint to the inclusion $L \hookrightarrow R$. In effect, an interior operator and a subframe carry the same information, and we will use them interchangeably.

Definition 3.3 Let RAlg be the category of Raney algebras and complete lattice homomorphisms satisfying $h(\Box a) \leq \Box h(a)$.

The next lemma is straightforward to prove.

Lemma 3.4 *Let $(R, \Box)$ and $(S, \Box)$ be Raney algebras, L the subframe of fixpoints of $(R, \Box)$, and K the subframe of fixpoints of $(S, \Box)$. For a complete lattice homomorphism $h: R \to S$ the following are equivalent:*

(1) *h is a morphism in* RAlg*;*
(2) $\Box h(\Box a) = h(\Box a)$*;*
(3) $h[L] \subseteq K$.

Moreover, h is an isomorphism in RAlg *iff it is a lattice isomorphism and* $h(\Box a) = \Box h(a)$.

We next come to the motivating example of Raney algebras. For a topological space X, we use $\mathcal{O}(X)$ to denote the frame of open sets of X. We note that $\mathcal{O}(X)$ is a subframe of $\mathsf{Up}(X)$, where X is considered as a poset under the specialization order.

Proposition 3.5 *For each topological space X, the pair* $(\mathsf{Up}(X), \Box)$ *consisting of the Raney lattice* $\mathsf{Up}(X)$ *of upsets of X under specialization and the restriction of the topological interior operator to* $\mathsf{Up}(X)$ *is a Raney algebra.*

Proof It is clear that $\mathcal{O}(X)$ is the $\Box$-fixpoints. Since $\mathsf{Up}(X)$ is a Raney lattice and each upset is a union of principal upsets $\uparrow x$, it is enough to show that $\uparrow x = \bigcap\{U \in \mathcal{O}(X) \mid x \in U\}$. But this follows from the definition of the specialization order. □

Our aim is to provide a contravariant adjunction between Top and RAlg which restricts to a dual equivalence between RAlg and the full subcategory Top_0 of Top. Our first step is the following.

Proposition 3.6 *There is a contravariant functor* $\mathcal{R}\colon \mathsf{Top} \to \mathsf{RAlg}$.

Proof For a topological space X, Proposition 3.5 yields that $\mathcal{R}(X) := (\mathsf{Up}(X), \Box)$ is a Raney algebra. If $f\colon X \to Y$ is continuous, then f is order-preserving with respect to the specialization orders on X and Y, so $f^{-1}\colon \mathsf{Up}(Y) \to \mathsf{Up}(X)$ is a well-defined complete lattice homomorphism, and $f^{-1}[\mathcal{O}(Y)] \subseteq \mathcal{O}(X)$. Thus, by Lemma 3.4, $\mathcal{R}(f) := f^{-1}$ is a morphism in RAlg, and it is easy to see that this defines a contravariant functor $\mathcal{R}\colon \mathsf{Top} \to \mathsf{RAlg}$. □

To provide a contravariant functor in the other direction, we recall a few facts discussed in Sect. 2. For a Raney lattice R, X_R is the poset of completely join prime elements of R, ordered by $x \sqsubseteq y$ iff $y \leq x$, and there is an isomorphism $\eta_R\colon R \to \mathsf{Up}(X_R)$. Also, for a complete lattice homomorphism $h\colon R \to S$ between Raney lattices, $h_*\colon S \to R$ is its left adjoint, and the restriction of h_* is a well-defined order-preserving map from X_S to X_R.

Proposition 3.7 *Let* $A = (R, \Box)$ *be a Raney algebra and L the* $\Box$*-fixpoints. Then* $\tau_L := \eta_R[L]$ *is a topology on* X_R *whose specialization order is* $\sqsubseteq$.

Proof Since L is a subframe of R, its image $\eta_R[L]$ is a subframe of $\mathsf{Up}(X_R)$, hence is a topology τ_L on X_R. Let $\leq_\tau$ be the specialization order of τ_L. As L is dense in R, each $x \in X_R$ is the meet of $\uparrow x \cap L$. Therefore, for $x, y \in X_R$, we have

$$\begin{aligned} x \leq_\tau y &\Leftrightarrow (\forall a \in L)(x \in \eta_R(a) \Rightarrow y \in \eta_R(a)) \\ &\Leftrightarrow (\forall a \in L)(x \leq a \Rightarrow y \leq a) \\ &\Leftrightarrow y \leq x \\ &\Leftrightarrow x \sqsubseteq y. \end{aligned}$$

□

Proposition 3.8 *There is a contravariant functor* $\mathcal{S}\colon \mathsf{RAlg} \to \mathsf{Top}$.

Proof Let $(R, \Box)$ be a Raney algebra and L the $\Box$-fixpoints. We set $\mathcal{S}(R, \Box) = (X_R, \tau_L)$. By Proposition 3.7, $\mathcal{S}(R, \Box) \in \mathsf{Top}$. For a morphism $h\colon R \to S$ of Raney algebras, we show that

$h_*\colon X_S \to X_R$ is continuous. For this it is sufficient to show that $h_*^{-1}(\eta_R(a)) = \eta_S(h(a))$ for each $a \in L$. Let $x \in X_S$. Since h_* is left adjoint to h, we have

$$x \in h_*^{-1}(\eta_R(a)) \text{ iff } h_*(x) \in \eta_R(a) \text{ iff } h_*(x) \leq a \text{ iff } x \leq h(a) \text{ iff } x \in \eta_S(h(a)).$$

Thus, h_* is continuous, and it is easy to see that setting $\mathcal{S}(h) = h_*$ defines a contravariant functor $\mathcal{S}\colon \mathsf{RAlg} \to \mathsf{Top}$. □

Lemma 3.9 *For a topological space X there is a continuous map $\varepsilon_X\colon X \to \mathcal{SR}(X)$ given by $\varepsilon_X(x) = {\uparrow}x$ for each $x \in X$. The maps ε_X provide a natural transformation $\varepsilon\colon 1_{\mathsf{Top}} \to \mathcal{SR}$.*

Proof Since ${\uparrow}x$ is a principal upset of the specialization order, it is completely join prime in $\mathsf{Up}(X)$, hence an element of $\mathcal{SR}(X)$. Therefore, ε_X is well defined. To see that it is continuous, it is enough to show that $\varepsilon_X^{-1}(\eta_{\mathsf{Up}(X)}(U)) = U$ for each U open in X. We have

$$x \in \varepsilon_X^{-1}(\eta_{\mathsf{Up}(X)}(U)) \text{ iff } \varepsilon_X(x) \in \eta_{\mathsf{Up}(X)}(U) \text{ iff } \varepsilon_X(x) \subseteq U \text{ iff } {\uparrow}x \subseteq U \text{ iff } x \in U.$$

To see that $\varepsilon\colon 1_{\mathsf{Top}} \to \mathcal{SR}$ is a natural transformation, let X and Y be topological spaces and $f\colon X \to Y$ a continuous map. We must show that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ \mathcal{SR}(X) & \xrightarrow[\mathcal{SR}(f)]{} & \mathcal{SR}(Y) \end{array}$$

Let $x \in X$. We have $\mathcal{SR}(f)(\varepsilon_X(x)) = \mathcal{SR}(f)({\uparrow}x) = (f^{-1})_*({\uparrow}x)$, which is the smallest upset A with ${\uparrow}x \subseteq f^{-1}(A)$. Since f is order-preserving, ${\uparrow}f(x) = \varepsilon_Y f(x)$ is the smallest such upset. Thus, $\mathcal{SR}(f)(\varepsilon_X(x)) = \varepsilon_Y f(x)$. □

Lemma 3.10 *For a Raney algebra $A = (R, \Box)$ the map $\eta_R\colon A \to \mathcal{RS}(A)$ is an isomorphism in* RAlg *that we call η_A. The maps η_A provide a natural isomorphism $\eta\colon 1_{\mathsf{RAlg}} \to \mathcal{RS}$.*

Proof For a Raney lattice R we have that $\eta_R : R \to \mathsf{Up}(X_R)$ is an isomorphism (see Theorem 2.4). For a Raney algebra $A = (R, \Box)$ we have $\mathcal{S}(A)$ is the space X_R whose opens are $\eta_R[L]$. Thus, $\mathcal{RS}(A) = (\mathsf{Up}(X_R), \Box')$ where $\Box'$ is the restriction of the interior operator on $\mathcal{S}(A)$ to the upsets. Since η_R is an isomorphism from the $\Box$-fixpoints to the $\Box'$-fixpoints, it is an isomorphism in RAlg by Lemma 3.4. It remains to see that $\eta\colon 1_{\mathsf{RAlg}} \to \mathcal{RS}$ is a natural transformation. Let $A = (R, \Box)$ and $B = (S, \Box)$ be Raney algebras and h a morphism between them. We must show that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\quad h \quad} & B \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ \mathcal{RS}(A) & \xrightarrow[\mathcal{RS}(h)]{} & \mathcal{RS}(B) \end{array}$$

For this it is sufficient to show that for each $a \in R$ we have $h_*^{-1}\eta_R(a) = \eta_S h(a)$, which follows from the proof of Proposition 3.8. □

Theorem 3.11 *$(\mathcal{R}, \mathcal{S})$ is a contravariant adjunction between* Top *and* RAlg.

Proof We must show for each space X that $R(\varepsilon_X) \circ \eta_{\mathcal{R}(X)}$ is the identity on $\mathcal{R}(X)$, and for each Raney algebra $A = (R, \Box)$ that $\mathcal{S}(\eta_A) \circ \varepsilon_{\mathcal{S}(A)}$ is the identity on $\mathcal{S}(A)$ (see [8, p. 81, Thm. 2] and adapt for contravariant adjunction).

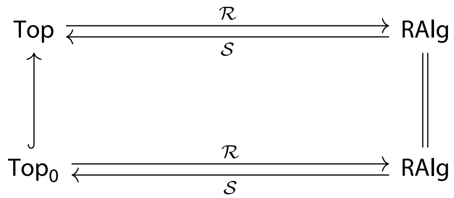
Note that $R(\varepsilon_X) \circ \eta_{\mathcal{R}(X)} = \varepsilon_X^{-1} \circ \eta_{\mathsf{Up}(X)}$. Let $x \in X$ and $U \in \mathsf{Up}(X)$. Then

$$\begin{array}{lll} x \in \varepsilon_X^{-1}(\eta_{\mathsf{Up}(X)})(U) & \text{iff} & \varepsilon_X(x) \in \eta_{\mathsf{Up}(X)}(U) \\ & \text{iff} & {\uparrow}x \in \eta_{\mathsf{Up}(X)}(U) \\ & \text{iff} & {\uparrow}x \subseteq U \\ & \text{iff} & x \in U. \end{array}$$

Thus, $\varepsilon_X^{-1}(\eta_{\mathsf{Up}(X)})(U) = U$.

Note that $\mathcal{S}(\eta_A) \circ \varepsilon_{\mathcal{S}(A)} = (\eta_R)_* \circ \varepsilon_{X_R}$. Let $x \in X_R$. Then $(\eta_R)_*(\varepsilon_{X_R}(x)) = (\eta_R)_*({\uparrow}x)$, which is the least element a of R with ${\uparrow}x \subseteq \eta_R(a)$. Since the order on X_R is the dual of the order on R, this is the least element a of R with ${\downarrow}x \subseteq \eta_R(a) = \{y \in X_R \mid y \leq a\}$, and hence is equal to x. Thus, $(\eta_R)_*(\varepsilon_{X_R}(x)) = x$. □

Theorem 3.12 *The contravariant adjunction $(\mathcal{R}, \mathcal{S})$ between* Top *and* RAlg *restricts to a dual equivalence between* RAlg *and the full subcategory* Top_0 *of* Top *consisting of T_0-spaces.*



Proof In Lemma 3.10 we showed that $\eta_A \colon A \to \mathcal{RS}(A)$ is an isomorphism in RAlg for each Raney algebra A. Therefore, it is enough to show that for each topological space X, the map $\varepsilon_X \colon X \to \mathcal{SR}(X)$ is a homeomorphism iff X is a T_0-space. Since $\mathcal{SR}(X) = \{{\uparrow}x \mid x \in X\}$, it is clear that ε_X is always onto. In Lemma 3.9 we showed that $\varepsilon_X^{-1}(\eta_{\mathsf{Up}(X)}(U)) = U$ for each open $U \subseteq X$. Since ε_X is onto, it follows that $\varepsilon_X(U) = \eta_{\mathsf{Up}(X)}(U)$. Therefore, ε_X is always onto, continuous, and open. Since X is a T_0-space iff the specialization order is a partial order, ε_X is one-to-one iff X is a T_0-space. The result follows. □

Putting Theorems 1.1, 1.2, and 3.12 together yields:

Corollary 3.13 *There is an adjunction between* RAlg *and* MT *that restricts to an equivalence between* RAlg *and* MT_0.

Remark 3.14 The above adjunction and equivalence can be realized directly by the functors $\mathcal{F} \colon \mathsf{RAlg} \to \mathsf{MT}$ and $\mathcal{G} \colon \mathsf{MT} \to \mathsf{RAlg}$ given as follows. The functor $\mathcal{F}$ sends a Raney algebra $(R, \Box)$ to $(B, \Box^+)$ where B is the MacNeille completion of the free Boolean extension of R, and $\Box^+ x = \bigwedge\{\Box a \mid x \leq a \in R\}$. The functor $\mathcal{G}$ sends $(B, \Box) \in \mathsf{MT}$ to the Raney algebra $(R, \Box)$ where R is the complete sublattice of B completely generated by the $\Box$-fixpoints of B, and $\Box$ on R is the restriction of $\Box$ on B.

Remark 3.15 One might consider generalizing the definition of Raney algebras by removing the condition that the $\Box$-fixpoints are dense. One still obtains a functor to topological spaces along the same lines as above, but this is no longer part of an adjunction with Top. To see this, consider the 4-element Boolean algebra B with $\Box$ defined so that its fixpoints are the bounds 0, 1. Note that the box fixpoints now fail to be dense. Then for $A = (B, \Box)$ we have $\mathcal{S}(A)$ is a 2-element trivial space, that $\mathcal{RS}(A)$ is the 2-element Raney algebra, and that $\mathcal{SRS}(A)$ is a 1-element space. So there can be no natural transformation η_A in this setting with $\mathcal{S}(\eta_A) \circ \varepsilon_{\mathcal{S}(A)}$ being the identity on $\mathcal{S}(A)$.

4 Further Remarks

To begin, as we saw in Lemma 3.10, the composite $\mathcal{RS}$ is naturally isomorphic to the identity functor on RAlg. The other composite $\mathcal{SR}$ provides an alternate path to the well-known fact that Top_0 is a reflective subcategory of Top.

Theorem 4.1 *$\mathcal{SR}$ is the T_0-reflection from* Top *to* Top_0.

Proof By [2, p. 28, Thm. 2] it is enough to show that for each $X \in \mathsf{Top}$, $Y \in \mathsf{Top}_0$, and a continuous map $f: X \to Y$ there is a unique continuous map $g: \mathcal{SR}(X) \to Y$ such that $g \circ \varepsilon_X = f$. Since $Y \in \mathsf{Top}_0$, it follows from Theorem 3.12 that ε_Y is a homeomorphism. Set $g = \varepsilon_Y^{-1} \circ \mathcal{SR}(f)$. Clearly g is continuous, and $g \circ \varepsilon_X = f$ by the naturality of ε given in Lemma 3.9. Since each ε_X is onto and hence an epimorphism in Top, the uniqueness follows. □

The dual equivalence between Top_0 and RAlg restricts to a dual equivalence between Top_1 and a full subcategory of RAlg. The following result shows that this full subcategory consists exactly of those Raney algebras $(R, \Box)$ where R is Boolean, and hence is exactly MT_1.

Theorem 4.2 *For a Raney algebra $A = (R, \Box)$, the dual space $\mathcal{S}(A)$ is T_1 iff R is a Boolean algebra.*

Proof First suppose R is Boolean. Then X_R is the set of atoms of R. Suppose $x, y \in X_R$ are distinct. By density, there is a $\Box$-fixpoint a such that $x \leq a$ and $y \not\leq a$. Therefore, $x \in \eta_R(a)$ and $y \notin \eta_R(a)$. Since $\eta_R(a)$ is open, this proves that $\mathcal{S}(A)$ is T_1. Conversely, if $\mathcal{S}(A)$ is T_1, then the specialization order is the identity. Therefore, $\mathcal{RS}(A)$ is the full powerset, hence a Boolean algebra. Since $A = (R, \Box)$ is isomorphic to $\mathcal{RS}(A)$, we conclude that R is a Boolean algebra. □

Corollary 4.3 MT_1 *is the full subcategory of* RAlg *consisting of those Raney algebras $(R, \Box)$ where R is a Boolean algebra.*

The full subcategory of RAlg that corresponds to Alexandroff T_0-spaces also has an easy description.

Theorem 4.4 *For a Raney algebra $A = (R, \Box)$, the dual space $\mathcal{S}(A)$ is Alexandroff iff $\Box$ is the identity.*

Proof Let L be the $\Box$-fixpoints of R. If $\Box$ is the identity, then $L = R$, so $\tau_L = \eta_R[R]$. Since $\eta_R[R]$ is the upsets of X_R, we conclude that $\mathcal{S}(A)$ is Alexandroff. Conversely, if $\mathcal{S}(A)$ is Alexandroff, then $\Box$ on $\mathcal{RS}(A)$ is identity because every upset of $\mathcal{S}(A)$ is open. Since $\mathcal{RS}(A)$ is isomorphic to A, we have that $\Box$ on A is also identity. □

Remark 4.5 The category of Alexandroff T_0-spaces is isomorphic to Pos. The above result yields that the category of Raney algebras where $\Box$ is identity is dually equivalent to Pos. This is as expected since this category is isomorphic to Ran (see Theorem 2.4).

We recall that a topological space X is *sober* if each closed irreducible set is the closure of a unique point. We are able to characterize those Raney algebras A with $\mathcal{S}(A)$ sober, however the characterization is largely a reformulation of ideas from pointfree topology. In doing so, we are led to a more natural condition on Raney algebras which corresponds to a weakening of the notion of sober spaces. We begin with the following standard notion from pointfree topology.

Definition 4.6 A filter F of a frame L is *completely prime* if for each $S \subseteq L$, from $\bigvee S \in F$ it follows that $F \cap S \neq \varnothing$.

It is easily seen that for a completely prime filter F of a frame L, there is a largest element of L that does not belong to F, and this largest element is meet prime. If X is a topological space, and $x \in X$, then $F_x = \{V \mid x \in V\}$ is a completely prime filter of the frame $\mathcal{O}(X)$. If X is sober the converse is also true as the following shows.

Lemma 4.7 *For a topological space X, the following are equivalent:*

(1) *X is sober.*
(2) *For each meet prime element $U \in \mathcal{O}(X)$ there is a unique $x \in X$ such that $U = -{\downarrow}x$.*
(3) *For each completely prime filter F of $\mathcal{O}(X)$ there is a unique $x \in X$ such that $F = F_x$.*

Proof The equivalence of (1) and (2) is easily seen by passing to complements and noting that ${\downarrow}x$ is the closure of x. For the equivalence of (2) and (3) see [10, Prop. 1.3.1]. □

Let $A = (R, \Box)$ be a Raney algebra with L its $\Box$-fixpoints. For each $x \in X_R$ let $L_x = {\uparrow}x \cap L$. Since $x \in X_R$ is a completely join prime element of R and L is a subframe of R, it is easily seen that each L_x is a completely prime filter of L.

Theorem 4.8 *Let $A = (R, \Box)$ be a Raney algebra with L its $\Box$-fixpoints. Then the dual space $\mathcal{S}(A)$ is sober iff each completely prime filter of L is equal to L_x for some necessarily unique $x \in X_R$.*

Proof We have that $\eta_R \colon L \to \mathcal{O}(\mathcal{S}(A))$ is an isomorphism. Therefore, for each $U \in \mathcal{O}(\mathcal{S}(A))$ there is a unique $a \in L$ with $U = \eta_R(a)$. Thus, for $x \in X_R$, we have

$$U \in F_x \text{ iff } x \in U \text{ iff } x \in \eta_R(a) \text{ iff } x \leq a \text{ iff } a \in L_x.$$

Consequently, $\eta_R[L_x] = F_x$. The result then follows from Lemma 4.7. □

Let $\mathsf{RAlg_S}$ be the full subcategory of RAlg consisting of those Raney algebras that satisfy the condition of Theorem 4.8 and let Sob be the full subcategory of $\mathsf{Top_0}$ consisting of sober spaces. Then the following is a direct consequence of Theorem 4.8.

Corollary 4.9 *The dual equivalence of Theorem 3.12 between RAlg and $\mathsf{Top_0}$ restricts to a dual equivalence between $\mathsf{RAlg_S}$ and Sob.*

We recall that a frame L is *spatial* if for $a, b \in L$, from $a \not\leq b$ it follows that there is a completely prime filter F of L containing a and missing b. Let Frm be the category of frames and frame homomorphisms (maps preserving finite meets and arbitrary joins), and let SFrm be its full subcategory of spatial frames. It is well-known (see, e.g., [10, Secs. II.4–6]) that there is a dual adjunction between Frm and Top, which restricts to a dual equivalence between SFrm and Sob. The following is then immediate.

Corollary 4.10 *There is an equivalence between RAlg_S and SFrm.*

Remark 4.11 The equivalence of Corollary 4.10 can be obtained by sending $(R, \Box) \in \mathsf{RAlg_S}$ to the spatial frame L of the $\Box$-fixpoints, and $L \in \mathsf{SFrm}$ to the Raney algebra $(\mathsf{Up}(X_L), \Box)$ where X_L is the poset of completely prime filters of L ordered by inclusion. The topology on X_L is given by $\{\varphi_L(a) \mid a \in L\}$ where $\varphi_L(a) = \{x \in X_L \mid a \in x\}$. The operation $\Box$ on $\mathsf{Up}(X_L)$ is the interior operator of this topology and is given by

$$\Box U = \bigcup \{\varphi_L(a) \mid \varphi_L(a) \subseteq U\}$$

for each $U \in \mathsf{Up}(X_L)$. Finally, we note that there is an interesting characterization of the Raney lattice $\mathsf{Up}(X_L)$ constructed from a frame L: the forgetful functor from Ran to Frm has a left adjoint that takes a frame L to $\mathsf{Up}(X_L)$.

Our characterization of sobriety in Theorem 4.8 suggests a natural order-theoretic condition on a Raney algebra $A = (R, \Box)$ that the meet of each completely prime filter of its $\Box$-fixpoints is completely join prime in R. This translates directly into the topological setting as follows.

Definition 4.12 A topological space X is *almost sober* if for each completely prime filter F of $\mathcal{O}(X)$ there is a unique $x \in X$ such that $\bigcap F = \uparrow x$.

Proposition 4.13 *Each sober space is almost sober, and each almost sober space is T_0.*

Proof We first observe that in any topological space X we have $\bigcap F_x = \uparrow x$ for each $x \in X$. Let $U \in F_x$. Then $x \in U$, so $\uparrow x \subseteq U$, and hence $\uparrow x \subseteq \bigcap F_x$. Suppose $y \notin \uparrow x$. Then there is $U \in \mathcal{O}(X)$ with $x \in U$ and $y \notin U$. Therefore, $U \in F_x$ and $y \notin U$. Thus, $y \notin \bigcap F_x$.

Next suppose that X is a sober space and F is a completely prime filter of $\mathcal{O}(X)$. Since X is sober $F = F_x$ for a unique $x \in X$ by Lemma 4.7. Therefore, $\bigcap F = \bigcap F_x = \uparrow x$. This proves that every sober space is almost sober. Finally, to see that an almost sober space X is T_0 suppose $x, y \in X$. If x, y are not distinguishable by open sets, then $\uparrow x = \uparrow y$. Therefore, $\bigcap F_x = \uparrow x = \uparrow y$. Since F_x is a completely prime filter of $\mathcal{O}(X)$ and X is almost sober, we must have $x = y$. Thus, X is T_0. □

We next provide several examples to show that these containments are strict.

Example 4.14 We give an example of a T_0-space that is not almost sober. Consider the natural numbers ω with the usual order and its Alexandroff topology. Then the open sets of this space consist of the empty set and the sets $\uparrow n$ for some $n \in \omega$. This space is T_0 since it is the Alexandroff topology of a poset. However, it is not almost sober since the collection F of all nonempty open sets is a completely prime filter of the frame of opens with $\bigcap F = \emptyset$.

Example 4.15 We give an example of an almost sober space that is not sober. Consider the ordinal $\omega + 1$ under its Alexandroff topology. This is not a sober space since this poset has an infinite ascending chain and sober is equivalent to Noetherian for Alexandroff spaces of posets. However, it is almost sober. To see this note that the frame of opens of this space is the chain whose least element is $\emptyset$, followed by the atom $\{\omega\}$, and then the sets $\uparrow n$ for $n \in \omega$. The completely prime filters of the frame of opens are the principal filters generated by the sets $\uparrow \alpha$ for $\alpha \leq \omega$, whose intersection is $\uparrow \alpha$, and the filter $F = \{\uparrow n : n \in \omega\}$, whose intersection is $\uparrow \omega$.

We conclude the paper with the following characterization of almost sober spaces.

Theorem 4.16 *A topological space X is almost sober iff each irreducible closed set C has a join in the specialization order of X.*

Proof Suppose that X is almost sober and C is an irreducible closed set of X. Consider $F_C = \{U \in \mathcal{O}(X) \mid U \cap C \neq \emptyset\}$. Then F_C is a completely prime filter of $\mathcal{O}(X)$ with $-C$ the largest open set not belonging to F_C. Since X is almost sober, there is a unique x such that $\bigcap F_C = \uparrow x$. Recall that $x \leq y$ iff every open set containing x contains y, that is, iff $y \in \bigcap F_x$. Thus, y is an upper bound of C iff $y \in \bigcap F_c$ for each $c \in C$. Since $F_C = \bigcup\{F_c : c \in C\}$, this occurs iff $y \in \bigcap F_C = \uparrow x$. It follows that $x = \bigvee C$.

Suppose that each irreducible closed set C has a join in the specialization order of X. Let F be a completely prime filter of $\mathcal{O}(X)$ and suppose that V is the largest open set that is not in F. Set $C = -V$. Note that C is an irreducible closed set and that F is equal to the filter F_C of open sets that intersect C nontrivially. Our assumption gives that C has a join in the specialization order. Let $x = \bigvee C$. Then $y \in \uparrow x$ iff y is an upper bound of C. By the argument in the first paragraph, this occurs iff $y \in \bigcap F_C$. As $F = F_C$, it follows that $\bigcap F = \uparrow x$. □

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